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LSZ-reduction, resonances and non-diagonal propagators: Gauge fields

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Abstract

We analyze in the Landau gauge mixing of bosonic fields in gauge theories with exact and spontaneously broken symmetries, extending to this case the Lehmann–Symanzik–Zimmermann (LSZ) formalism of the asymptotic fields. Factorization of residues of poles (at real and complex values of the variable p^2) is demonstrated and a simple practical prescription for finding the "square-rooted" residues, necessary for calculating S-matrix elements, is given. The pseudo-Fock space of asymptotic (in the LSZ sense) states is explicitly constructed and its BRST-cohomological structure is elucidated. Usefulness of these general results, obtained by investigating the relevant set of Slavnov–Taylor identities, is illustrated on the one-loop examples of the Z^0-photon mixing in the Standard Model and the G_Z-Majoron mixing in the singlet Majoron model.

1. Introduction

Mixing of fields is a common feature of many quantum field theory models. For example, scalar fields mix in many extended models of the Higgs sector of the Standard Electroweak Theory; already in the Standard Model (SM) one has to do with mixing of vector fields (the B_μ and the W^3_μ fields – known as the photon-Z^0 mixing) and with mixing of fermions (the Cabibbo–Kobayashi–Maskawa mixing), see e.g. [1]. In tree level calculations, the mixing is re-

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moved by appropriate redefinitions of the fields but in higher orders it reappears and extraction of S-matrix elements from Green's functions requires addressing this problem. Moreover, some particle states identified at the tree level become, when the loop corrections are included, unstable (resonances) and the structure of the Fock space of true asymptotic states of the model is usually (even in the perturbative expansion) different than the Fock space of the corresponding non-interacting theory.

In general, the proper way of extracting S-matrix elements is provided by the Lehmann–Symanzik–Zimmermann (LSZ) asymptotic approach which basically consist of analyzing the pole structure of the relevant two-point functions of the fields which mix, and reconstructing on this basis the Fock space of the true asymptotic states. Yet, to the best of our knowledge, in the case of field mixing in general gauge theories this has never been analyzed in details.

In the simplest case of mixing of several scalar fields ϕ^i (which we take to be real, that is Hermitian operators) the (connected) two point Green's function (propagator) can in general be written in the form

$$\langle T(\phi^k(x)\phi^j(y))\rangle = \int \frac{\mathrm{d}^4 p}{(2\pi)^4}\, e^{-i\,p\cdot(x-y)} \left\{ \sum_{\ell} \zeta^k_{S[\ell]}\, \frac{i}{p^2 - m^2_{S(\ell)}}\, \zeta^j_{S[\ell]} + \begin{bmatrix}\text{non-pole}\\ \text{part}\end{bmatrix} \right\}. \quad (1)$$

Factorization of the residues of poles, some of which occur at real and other at complex values of p^2, is a well-known property [2].[1] The factors $\zeta^k_{S[\ell]}$ associated with poles at real values $m^2_{S(\ell)}$ of p^2 are crucial for obtaining correctly normalized (i.e. consistent with unitarity) transition amplitudes between initial and final states involving stable particles. More precisely, the Cutkosky–Veltman rules guarantee [5] that the S-matrix is unitary provided *i)* asymptotic (free) fields appearing in the LSZ-reduction formula for the S-operator (see e.g. [6] and the formula (13) below) are normalized so as to reproduce the behavior of the corresponding (full) two-point functions near the poles associated with stable particles, and *ii)* poles at complex values of p^2 are associated with no asymptotic states (fields), i.e. unstable particles contribute to the S-matrix only through the internal lines. Thus, the asymptotic field $\boldsymbol{\phi}^j$ associated with ϕ^j has the form $\boldsymbol{\phi}^j = \sum'_{\ell} \zeta^j_{S[\ell]}\Phi^\ell$, where Φ^ℓ are canonically normalized free scalar fields constructed out of the annihilation and creation operators of a spin 0 particle of mass $m_{S(\ell)}$, and the summation runs over indices ℓ labeling only real poles (this is indicated by the prime).

The first LSZ based extraction of S-matrix elements (disregarding the unstable character of the particles) in the presence of mixing of two scalar fields (neutral CP even components of the two Higgs doublets of the Minimal Supersymmetric Standard Model) was presented in [7]. The mixing of W^3_μ and B_μ gauge fields of the Standard Model was studied in this framework in [8], and factorization of residues of the pole at $p^2 = 0$ (corresponding to the photon) and at a complex value of p^2 (corresponding to the unstable Z^0 boson) was explicitly demonstrated. The LSZ approach to fermionic mixing was presented first in the context of leptogenesis in [9,10] where factorization of residues of the poles corresponding to unstable Majorana neutrinos was demonstrated (see also [11,12]). While asymptotic (in and out) particle states corresponding to poles in (1) at complex values of $m^2_{S(\ell)}$ do not, strictly speaking, exist, the factors $\zeta^j_{S[\ell]}$ associated with such poles are, nevertheless, useful in studying properties of resonances as shown in [2,8–10].

[1] Factorization of residues at real poles follows also from formal manipulations [3,4] that is, from inserting the complete set of asymptotic states between the field operators in the left hand side of (1). The $\zeta^k_{S[\ell]}$ factors are then simply equal $\langle 0|\phi^k(0)|\mathbf{p},\ell\rangle$, where $|0\rangle$ is the true vacuum of the theory and $|\mathbf{p},\ell\rangle$ are the states of a single spin 0 particles labeled by ℓ.

Clearly, treatment of mixing of scalar fields is essential in studies of multifield Higgs sectors of various extensions of the SM. Similarly, the mixing of vector fields is a typical feature of theories (e.g. GUT models) based on gauge symmetry groups higher than the group $SU(3)_c \times SU(2)_L \times U(1)_Y$ of the SM. In view of the ubiquity of mixing of fields, the problem of formulating an optimal prescription for computing the coefficients $\zeta^k_{S[\ell]}$ parameterizing the residues in (1) have gained in recent years a renewed interest. Mixing of three scalar fields was analyzed only recently in the paper [13], in which the factorization property (1) was demonstrated and explicit formulae for the coefficients $\zeta^k_{S[\ell]}$ were given. The results were applied to the neutral Higgs sector of the MSSM; it was shown that cross-sections obtained neglecting the non-pole part in Eq. (1) agree to a good accuracy with the cross-sections based on the full propagators. Analysis of a generic mixing of n fermionic fields was recently given in [14–16].[2]

In [19] we have reconsidered general mixing of scalar and fermionic fields, simplifying and generalizing prescriptions for calculating the ζ factors available in the cited literature. In the case of fermions we have analyzed in details poles corresponding to the arbitrary system of Weyl fields, obtaining prescriptions for Dirac-type and Majorana-type spin 1/2 particles (or resonances) as special cases. Our approach is closest in spirit to the one of [14–16]. There are, however, some differences. Firstly, we followed the philosophy of keeping the renormalization scheme as general as possible. In particular, we did not impose any concrete renormalization conditions on the two-point functions. Second, we offered a technical improvement in comparison with the analyses of [2,14–16], where the cofactor matrix of one-particle irreducible two-point functions at the pole was used to get the formulae for ζ. In contrast, the factors ζ in our approach are expressed directly in terms of properly normalized eigenvectors of certain "mass-squared matrices", so that the case of degenerated eigenvalues is naturally covered by our prescription. Thus, the prescription for finding ζ proposed in [19] can be considered a direct generalization of the standard procedure for finding tree-level mass eigenstates.

The purpose of this paper is to extend the approach of [19] to the mixing of n vector fields in a general gauge field theory. The complication characteristic for gauge fields in general, and their mixing in particular, is the presence of unphysical degrees of freedom which contribute to residues of poles of the two-point functions but do not correspond to physical particles. To properly identify the ζ factors corresponding to physical particles (or resonances) we perform a careful analysis of the relevant set of Slavnov–Taylor identities and explicitly construct the asymptotic (in the LSZ sense) vector and scalar fields. We also demonstrate that, in case of generic mixing, the unphysical components of these asymptotic fields create (out of the vacuum) states which combine into the Kugo–Ojima quartet representations of the BRST algebra [20] what is essential for unitarity of the S-matrix [6,20].

We have decided to restrict this study to the Landau gauge, since this gauge offers some practical advantages: it is Lorentz covariant, renormalization group invariant and provides the simplest way of calculating the effective potential (see e.g. [21] for a recent determination of the three-loop effective potential of scalar fields in a general renormalizable model in the $\overline{\text{MS}}$ scheme). A dedicated analysis of the Landau gauge case is justified also at the technical level: firstly, the Nakanishi–Lautrup auxiliary fields cannot be integrated out in this gauge. Secondly, in the Landau gauge the would-be Goldstone bosons produce poles at $p^2 = 0$ in the propagators of system of scalar fields and a prescription is necessary to properly identify the associated ζ

[2] In the context of field mixing we should mention also the paper [17] in which the possibility of imposing on-shell renormalization conditions in systems with mixed scalar, vector and fermionic fields was studied. Mixing of fermions treated in this approach was re-examined in [18] with the aid of special parametrization of the propagator.

factors in situations where there are also poles at $p^2 = 0$ corresponding to physical massless Goldstone bosons of spontaneously broken global symmetries. It is here that our approach of Ref. [19], covering also the case of degeneracies, becomes particularly advantageous; combined with the additional symmetry of the Faddeev–Popov sector of the Landau gauge action [22], it allows for unambiguous identification of the $\zeta_{S[\ell]}$ factors corresponding to physical massless spin 0 particles.

Analysis of mixing of vector fields in a non-Landau R_ξ gauge requires in principle only a few minor changes[3] and will be given elsewhere.

The paper is organized as follows. In Section 2.1 we summarize (and reformulate in a slightly more convenient way) the prescription of [19] for extracting the factors $\zeta^{\,j}_{S[\ell]}$ from Green's functions of scalar fields. In Sec. 2.2 this prescription is directly generalized to the case of mixing of vector fields without presenting the detailed structure of the asymptotic fields, so that the reader interested in practical aspects of the procedure, that is in extracting the factors ζ_V necessary for computing elements of the S-matrix with spin 1 particles in asymptotic states, is not distracted by technicalities. The prescription for properly identifying the factors $\zeta^{\,j}_{S[\ell]}$ corresponding to physical Goldstone bosons is also given. To illustrate the main points on some examples we first give in Section 3 general one-loop formulae for all possible (in the Landau gauge) self-energies of the system of vector and scalar fields of a general renormalizable model in the $\overline{\text{MS}}$ scheme,[4] cultivating in this way the long tradition of providing ready-to-use general formulae, see e.g. [27], [28], [29], [30], [31], [21], [32], [33]. Using these general results we reconsider in Section 4.1 the Z-photon mixing from the point of view of the asymptotic LSZ approach. The problem of properly identifying the would-be Goldstone modes and the true Goldstone bosons is illustrated in Sec. 4.2 on the example of the Singlet Majoron Model [34]. The technicalities: the analysis of Slavnov–Taylor identities, construction of the asymptotic fields and related issues, which constitute in fact the main results of the paper, are relegated to Section 5.

We end this introduction by summarizing our notation and conventions. In most of formulae indices are suppressed and the matrix multiplication is understood. The summation convention is used only when an upper index is contracted with a lower one; whenever ambiguities may arise, sums are explicitly displayed. The Minkowski metric has the form

$$\eta = [\eta_{\mu\nu}] = \mathrm{diag}(+1, -1, -1, -1)\,.$$

Our convention for Fourier transform of fields is summarized by the formulae

$$\mathcal{F}(x) = \int \mathrm{d}^4 l\, e^{-ilx}\, \hat{\mathcal{F}}(l) \qquad \Rightarrow \qquad \frac{\delta}{\delta \mathcal{F}(x)} = \int \frac{\mathrm{d}^4 l}{(2\pi)^4}\, e^{ilx}\, \frac{\delta}{\delta \hat{\mathcal{F}}(l)}\,. \tag{2}$$

We assume that the model in question has already been renormalized in an arbitrary renormalization scheme consistent with the gauge symmetry. Thus, all fields and correlation functions are

[3] In general (non-Landau) R_ξ gauges vector fields mix beyond the tree-level with the scalar ones giving rise to non-vanishing mixed vector-scalar propagators [23–26]. Therefore the asymptotic vector fields can create/annihilate also physical spin 0 particles. One thus needs a prescription for an additional (component of) eigenvector ζ which determines the contribution of physical scalar field mode to the asymptotic vector field.

[4] General formulae for fermionic self energies are collected in [19]. All these formulae were obtained using the naive prescription for the γ^5 matrix.

considered as renormalized ones.[5] The (renormalized) one-particle irreducible (1PI) effective action Γ is the generating functional for renormalized 1PI Green's functions. For instance, the two-point function of scalar fields ϕ^j is given by

$$\frac{\delta}{\delta\hat{\phi}^j(p)}\frac{\delta}{\delta\hat{\phi}^k(p')}\Gamma[\phi,\ldots]\bigg|_0 = (2\pi)^4\delta^{(4)}(p'+p)\,\widetilde{\Gamma}_{kj}(p',p)\,. \tag{3}$$

The functional derivatives (which act always from the left) in (3) are taken at the "point" at which *all* fields vanish (this is indicated by the vertical bar with the subscript 0). In particular, we always assume that the scalar fields have already been shifted if necessary, so that they have vanishing vacuum expectation values (VEVs); in other words we assume that

$$\frac{\delta\Gamma[\phi,\ldots]}{\delta\phi^i(x)}\bigg|_0 = 0. \tag{4}$$

2. Practical prescriptions

2.1. Mixing of scalar fields

We start by recapitulating the prescription, formulated in [19], for the pole part of the propagator of a system of scalar fields $\{\phi^j\}$ (which, without loss of generality are assumed to be all Hermitian with vanishing VEVs). The 1PI two-point function of such a system of scalars is of the general form

$$\widetilde{\Gamma}_{kj}(-p,p) = \left[p^2\mathbb{1} - M_S^2(p^2)\right]_{kj}\,, \tag{5}$$

with a symmetric matrix $M_S^2(s) = M_S^2(s)^\top \equiv (M_S^{\text{tree}})^2 + \Sigma_S(p^2)$. Inverting the matrix $\widetilde{\Gamma}_{kj}(-p,p)$ we get the matrix of propagators

$$\widetilde{G}^{kj}(p,-p) = i\left[\left(p^2\mathbb{1} - M_S^2(p^2)\right)^{-1}\right]^{kj}\,. \tag{6}$$

The poles of (6) are at values $p^2 = m^2_{S(\ell)}$ which are solutions to the following equation

$$\det(s\mathbb{1} - M_S^2(s))\Big|_{s=m^2_{S(\ell)}} = 0\,. \tag{7}$$

Let the vectors $\zeta_{S[\ell_1]}, \zeta_{S[\ell_2]}, \ldots$, form a basis of the eigenspace of the matrix $M_S^2(m^2_{S(\ell)})$ corresponding to its eigenvalue[6] $m^2_{S(\ell)}$

$$M_S^2(m^2_{S(\ell)})\,\zeta_{S[\ell_r]} = m^2_{S(\ell)}\,\zeta_{S[\ell_r]}\,, \tag{8}$$

obeying the following normalization/orthogonality conditions

$$\zeta^\top_{S[\ell_r]}\left[\mathbb{1} - M_S^{2\prime}(m^2_{S(\ell)})\right]\zeta_{S[\ell_q]} = \delta_{rq}\,, \tag{9}$$

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[5] In particular, we assume that the finite counterterms have been adjusted, if necessary, so as to restore the Slavnov–Taylor identities for the gauge symmetry (see e.g. [35] for a discussion in the context of dimensional regularization with the consistent 't Hooft–Veltman–Breitenlohner–Maison prescription for γ^5).

[6] In this notation $m^2_{S(\ell)} \neq m^2_{S(\ell')}$ for $\ell \neq \ell'$. Notice that, in general, eigenvalues of the matrix $M_S^2(m^2_{S(\ell)})$ other than $m^2_{S(\ell)}$ are not solutions of (7) and are irrelevant for the problem of mixing.

in which $M^{2\prime}_S(s) \equiv \mathrm{d}M^2_S(s)/\mathrm{d}s$. As shown in [19], the propagator (6) takes then the form[7]

$$\widetilde{G}^{kj}(p,-p) = \sum_{\ell}\sum_{r} \zeta^{k}_{S[\ell_r]} \frac{i}{p^2 - m^2_{S(\ell)}} \zeta^{j}_{S[\ell_r]} + [\text{non-pole part}]\,. \tag{10}$$

Moreover, if Feynman integrals contributing to $M^2_S(p^2)$ do not acquire imaginary parts in a left neighborhood $\mathscr{U}_\ell \subset \mathbb{R}$ of $p^2 = (m^{\text{tree}}_{S(\ell)})^2$, so that the following reality condition

$$M^2_S(s) = M^2_S(s)^\star\,, \qquad \forall_{s\in\mathscr{U}_\ell}\,, \tag{11}$$

is satisfied, then all terms of a formal power series

$$m^2_{S(\ell)} = (m^{\text{tree}}_{S(\ell)})^2 + \mathcal{O}(\hbar)\,,$$

are real and there exist vectors $\zeta_{S[\ell_r]}$ obeying Eqs. (8)–(9) and such that $\zeta_{S[\ell_r]} = \zeta^\star_{S[\ell_r]}$ for all r.

Some comments are in order. The normalization conditions (9) have here a different form than the ones given in [19] but are, nevertheless, equivalent to them; their form (9) will be more convenient in what follows. The left hand side of Eq. (9) is symmetric in the indices r and q. Therefore starting with an arbitrary basis of the eigenspace, say a set of vectors $\{\xi_{[\ell_r]}\}$, one can construct vectors obeying Eq. (9) provided $\xi_{[\ell_r]}$ are in a one-to-one correspondence with the eigenvectors of the tree-level mass matrix.[8]

The form (10) of the propagator uniquely determines the form

$$\boldsymbol{\phi}^j = \sum_{\ell}{}' \sum_{r} \zeta^{j}_{S[\ell_r]} \Phi^{\ell_r}\,, \tag{12}$$

of the asymptotic (in the LSZ sense) field corresponding to ϕ^j. The prime on the first sum in (12) indicates that it runs only over indices ℓ labeling poles of the propagator (10) located at real values of p^2 (we assume the corresponding vectors $\zeta_{S[\ell_r]}$ are chosen real). The operators Φ^{ℓ_r} in (12) are Hermitian scalar free field operators built out of the creation and annihilation operators of spin 0 particles of mass $m_{S(\ell)}$ acting in the standard way in the Fock space of the in (or out) states, and are such that one-particle states created by Φ^{ℓ_r} and by $\Phi^{\ell'_{r'}} \neq \Phi^{\ell_r}$ from the vacuum are orthogonal.[9] This form of (12) guarantees that (the Fourier transform of) $\langle 0|T\boldsymbol{\phi}^j(x)\boldsymbol{\phi}^j(y)|0\rangle$ reproduces the behavior of (6) in the vicinity of all poles located on the real axis. The asymptotic field $\boldsymbol{\phi}^j$ allows us to write the LSZ formula for the S-operator in a compact form [6]

$$S = :\!\exp\left\{-\int \mathrm{d}^4x\, \boldsymbol{\phi}^j(x) \int \mathrm{d}^4y\, \Gamma_{jk}(x,y) \frac{\delta}{\delta J_k(y)}\right\}\!: \exp(i\,W[J])\bigg|_{J=0}\,, \tag{13}$$

which when inserted between states of the asymptotic in (or out) Fock space yields S-matrix elements corresponding to transitions between stable particles. $\Gamma_{jk}(x,y)$ in (13) is the Fourier

[7] In order to ensure that the propagator takes on the simple form (10), one has to assume that each generalized eigenvector (see e.g. [36]) of $M^2_S(m^2_{S(\ell)})$ associated with the eigenvalue $m^2_{S(\ell)}$ is an ordinary eigenvector (that is, in the Jordan basis the block of the matrix $M^2_S(m^2_{S(\ell)})$ corresponding to its eigenvalue $m^2_{S(\ell)}$ is diagonal). There is no need to investigate other (unphysical) generalized eigenspaces of $M^2_S(m^2_{S(\ell)})$; in particular as a whole the matrix $M^2_S(m^2_{S(\ell)})$ can be even non-diagonalizable. We also assume that the derivative $M^{2\prime}_S(m^2_{S(\ell)})$ is infrared-finite, so that the singularities of the propagator are poles rather than branch points (see e.g. [37]).

[8] One could worry that the condition (9) cannot be imposed since e.g. $[1,\ i][1,\ i]^\top = 0$, however such a pathology is impossible at the tree-level, and thus it is impossible for the formal power series.

[9] For completeness, the explicit form of Φ^{ℓ_r} in our conventions is given is Sec. 5.5 below, cf. Eq. (155).

transform of (5) and the normal ordering refers to the free quantum fields $\boldsymbol{\phi}^j$. The functional $W[J]$ generating connected Greens functions is related through the Legendre transform to the (renormalized) 1PI effective action $\Gamma[\phi]$

$$\Gamma[\phi] = W[\mathcal{J}^\phi] - \int d^4x\, \mathcal{J}_j^\phi(x)\cdot\phi^j(x)\,, \qquad \left.\frac{\delta W[J]}{\delta J_j(x)}\right|_{J=\mathcal{J}^\phi} = \phi^j(x)\,. \tag{14}$$

In practical terms the formula (13) means that to obtain the correctly normalized (i.e. consistent with unitarity) amplitude of a process involving a particle corresponding to the field operator Φ^{ℓ_r}, the eigenvector $\zeta^j_{S[\ell_r]}$ has to be contracted with the appropriate amputated correlation function $\mathcal{A}_{j\ldots}(p,\ldots)$ of the scalar field ϕ^j evaluated at $p^2 = m^2_{S(\ell)}$.

As we already said, vectors $\zeta_{S[\ell_r]}$ corresponding to complex poles $m^2_{S(\ell)}$ in Eq. (10), even though they are not associated with asymptotic fields, are useful in the study of properties of unstable particles, as they govern the behavior of amplitudes for $s \approx \mathrm{Re}(m^2_{S(\ell)})$ [9,10] (see also [11,12]). In particular, the imaginary part of $\zeta_{S[\ell_r]}$ is one of the sources of CP-asymmetry in decays of unstable states [9].

We also note that the formulae (12) and (8)–(9) are obvious at the tree-level. In particular, $M_S^{2\,\prime}(p^2) = \mathcal{O}(\hbar)$ and therefore (12) is nothing but an expansion of the scalar field in an orthonormal basis of eigenvectors of the mass-squared matrix. From this point of view, Eq. (9) defines a "quantum-corrected metric" which fixes correct normalization of the eigenvectors in higher orders in $\hbar$.

2.2. *Vector and scalar fields*

We consider now a set $\{A^\alpha_\mu\}$ of (renormalized) Hermitian vector fields, together with a set $\{\phi^j\}$ of Hermitian scalar fields (having vanishing vacuum expectation values). In order to fix the conventions, we give here an expression for a covariant derivative of scalars

$$(D_\mu\phi)^j = \partial_\mu\phi^j + A^\alpha_\mu[\mathcal{T}_\alpha]^j{}_k(\phi^k + v^k)\,, \tag{15}$$

where $\mathcal{T}_\alpha$ are real antisymmetric generators of the gauge group in the representation formed by the scalars; they contain gauge couplings and satisfy the commutation relations $[\mathcal{T}_\alpha, \mathcal{T}_\beta] = \mathcal{T}_\gamma\, e^\gamma{}_{\alpha\beta}$ with real structure constants $e^\gamma{}_{\alpha\beta}$. As said, ϕ^j have vanishing VEVs; v^j are the VEVs of "fields in the symmetric phase" $\phi^j_{\mathrm{sym}} \equiv \phi^j + v^j$. Thus, v^j are determined by the condition that the complete tadpole of ϕ^j vanishes (cf. Eq. (4)), which gives v^j as a formal power series in $\hbar$[10]

$$v^j = v^j_{(0)} + \hbar\, v^j_{(1)} + \mathcal{O}(\hbar^2)\,. \tag{16}$$

[10] In order to simplify the notation, we have assumed in (15) that none of the components ϕ^j is a Stueckelberg field (see e.g. [38] and references therein). Nonetheless, everything what we say here works also in the presence of Stueckelberg scalars, provided one makes the replacement

$$\mathcal{T}_\alpha v \mapsto \mathcal{T}_\alpha v + \bar{P}_\alpha\,,$$

where coefficients $\bar{P}_\alpha$ obey $\mathcal{T}_\beta \bar{P}_\alpha = 0$ and (in a natural basis of the gauge Lie algebra) can be nonzero only for indices $\alpha = \alpha_A$ associated with the Abelian ideal.

The most general form of the renormalized[11] 1PI two-point function of vector fields is

$$\widetilde{\Gamma}^{\mu\nu}_{\alpha\beta}(-q,q) \equiv -\eta^{\mu\nu}\Big[q^2\mathbb{1} - M_V^2(q^2)\Big]_{\alpha\beta} + q^\mu q^\nu \mathscr{L}_{\alpha\beta}(q^2)\,. \tag{17}$$

The general form of the 1PI two-point function of scalar fields, which must be considered in parallel to that of vector fields, if some of gauge symmetries are broken by VEVs of scalars, is still given by (5). Even though the mixed vector-scalar two point function $\widetilde{\Gamma}^{\mu}_{\alpha j}(-q,q)$ is, in general, non-vanishing, the Landau gauge condition ensures that the mixed propagator vanishes

$$\widetilde{G}\,^{j\beta}_{\nu}(q,-q) = 0. \tag{18}$$

Thus, in addition to the propagator (6) of scalar fields, for practical purposes it suffices to consider the propagator of vector fields which takes (see Section 5) the form:

$$\widetilde{G}^{\beta\delta}_{\nu\rho}(q,-q) = -i\left[\eta_{\nu\rho} - \frac{q_\nu q_\rho}{q^2}\right]\left[\left(q^2\mathbb{1} - M_V^2(q^2)\right)^{-1}\right]^{\beta\delta}. \tag{19}$$

Since the “denominator” of (19) has the same structure as that of (5), one can immediately write

$$\left[\left(q^2\mathbb{1} - M_V^2(q^2)\right)^{-1}\right]^{\beta\delta} = \sum_\lambda \sum_r \zeta^{\beta}_{V[\lambda_r]}\,\frac{1}{q^2 - m^2_{V(\lambda)}}\,\zeta^{\delta}_{V[\lambda_r]} + [\text{non-pole part}]\,. \tag{20}$$

The complete pole part of the full propagator (19) will be given in Section 5 (the formula (99)). The formula (20) is however all one needs to write down those terms of the asymptotic vector field $\mathbf{A}^\alpha_\mu$ which are relevant for computing S-matrix amplitudes of processes with stable spin 1 particles in the initial and/or final states:

$$\mathbf{A}^\alpha_\mu = \sum_\lambda{}' \sum_r \zeta^{\alpha}_{V[\lambda_r]}\mathbb{A}^{\lambda_r}_\mu + \dots\,. \tag{21}$$

As in (12) the prime over the first sum indicates that it runs only over the indices λ labeling poles at real values $m^2_{V(\lambda)}$ of q^2. With each independent eigenvector $\zeta_{V[\lambda_r]}$ corresponding to such a pole associated is in (21) a free Hermitian vector field $\mathbb{A}^{\lambda_r}_\mu$ built out of the spin 1, mass $m_{V(\lambda)}$ particle annihilation and creation operators acting in the Fock space of the in (or out) states. The operator[12] $\mathbb{A}^{\lambda_r}_\mu$ has the unitarity gauge structure (if $m_{V(\lambda)} \neq 0$), or the Coulomb gauge structure (if $m_{V(\lambda)} = 0$). As in the case of the asymptotic field (12), the one-particle states created/annihilated by $\mathbb{A}^{\lambda_r}_\mu$ and by $\mathbb{A}^{\lambda'_{r'}}_\mu \neq \mathbb{A}^{\lambda_r}_\mu$ are orthogonal to each other. The ellipsis in (21) stand for free operators creating/annihilating in the in (or out) Fock space states belonging to Kugo–Ojima quartet representations [20]; the explicit formulae for these operators are given in Sec. 5.5. With all these operators taken into account the (Fourier transform of the) two point function $\langle 0|T(\mathbf{A}^\alpha_\mu(x)\mathbf{A}^\beta_\nu(y))|0\rangle$ reproduces the behavior of (19) near all poles located on the real axis. Using the asymptotic field (21) in the formula (13) for the S-operator[13] then shows that the amplitude of a process with a stable spin 1 particle corresponding to $\mathbb{A}^{\lambda_r}_\mu$ in the initial or final state is obtained by contracting the amputated correlation functions $\mathcal{A}^{\mu}_{\alpha\dots}(p,\dots)$ of fields

[11] Recall that we allow for completely arbitrary renormalization conditions that are consistent with Slavnov–Taylor identities, see e.g. [39].

[12] The explicit form of the operator $\mathbb{A}^{\lambda_r}_\mu$ in our conventions is given is Sec. 5.5 (the formula (134)).

[13] In our conventions, Eqs. (13)–(14) are valid in the generic case, provided that indices j and k run over all components of all fields, including vectors, fermions, (anti)ghosts and Nakanishi–Lautrup multipliers, see e.g. [6].

A^{α}_{μ} with the eigenvector $\zeta^{\alpha}_{V[\lambda_r]}$ and the appropriate (canonically normalized) polarization vector $e_{\mu}(\mathbf{p}, m_{V(\lambda)})$ or $e_{\mu}(\mathbf{p}, m_{V(\lambda)})^{\star}$.

In the presence of spontaneous breaking of some gauge symmetries, it is also necessary to identify those terms in the decomposition (12) of the asymptotic scalar field which create/annihilate physical states. This is particularly easy if there are no Goldstone bosons of spontaneously broken global symmetries[14] as then all fields Φ^{ℓ_r} corresponding to $m_{S(\ell)} = 0$ create would-be Goldstone bosons while all remaining fields are associated with physical particles. If the true Goldstone bosons are present (e.g. in the singlet Majoron model [34], see also Section 4.2), we need a prescription for identifying massless eigenvectors $\zeta_{S[\ell_r]}$ associated with them. The gauge symmetry implies that the eigenvectors $\zeta_{S[\ell_r]}$ corresponding to the would-be Goldstone bosons are linear combinations of vectors[15] $\mathcal{T}_{\alpha} v$. The orthogonality condition (9) then suggests that of all vectors $\zeta_{S[\ell_r]}$ associated with poles at $p^2 = 0$, to physical massless states should correspond vectors $\zeta_{S[\ell_r]}$ such that

$$\zeta^{\top}_{S[\ell_r]} \left[\mathbb{1} - M^{2\,\prime}_{S}(0) \right] \mathcal{T}_{\alpha} v = 0\,, \tag{22}$$

for all indices α.[16] In Sec. 5.5 we will show that the states of the asymptotic Fock spaces associated with eigenvectors $\zeta_{S[\ell_r]}$ obeying this condition do indeed belong to the physical subspace of the kernel of BRST charge.

It should be also stressed that the normalization condition, which for vectors $\zeta^{j}_{S[\ell]}$ takes the form (9), has to be slightly modified in order to avoid *spurious* infrared divergences. Take, for instance, the Z-photon block of the Standard Model (SM, see e.g. [1]); the 2-by-2 matrix $M^{2\,\prime}_{V}(0)$ (more precisely, its ZZ entry) is IR divergent at one-loop order, however the photonic singularity is still a pole. The IR-finiteness of the whole matrix $M^{2\,\prime}_{V}(0)$ is therefore too strong a requirement. In Sec. 5.3 we will show that Eq. (20) holds provided $M^2_V(s)$ is continuous at each $m^2_{V(\lambda)}$ and that the limit

$$\lim_{q^2 \to m^2_{V(\lambda)}} \left\{ M^{2\,\prime}_{V}(q^2)\, \xi \right\}\,, \tag{23}$$

exists for each ξ belonging to the eigenspace[17] $M^2_V(m^2_{V(\lambda)})$ associated with $m^2_{V(\lambda)}$.

The vectors $\zeta_{V[\lambda_r]}$ appearing in (20) are then elements of a basis of the eigenspace

$$M^2_V(m^2_{V(\lambda)})\, \zeta_{V[\lambda_r]} = m^2_{V(\lambda)}\, \zeta_{V[\lambda_r]}\,, \tag{24}$$

obeying the normalization conditions

$$\lim_{q^2 \to m^2_{V(\lambda)}} \left\{ \zeta^{\top}_{V[\lambda_r]} \left[\mathbb{1} - M^{2\,\prime}_{V}(q^2) \right] \zeta_{V[\lambda_t]} \right\} = \delta_{rt}\,. \tag{25}$$

Furthermore, if *massless spin 1* particles are present, an additional assumption is necessary to ensure that the singularity of the full propagator (19) at $q^2 = 0$ is a (second order) pole: the limit

[14] Of physical spin 0 particles, only Goldstone bosons can naturally be massless.

[15] More precisely, this fact follows from the "non-renormalization theorem" expressed by the relation (87), which is a manifestation of an additional symmetry of the action specific for the Landau gauge [22].

[16] At the tree-level this reduces to a well-known condition (see e.g. (1.1) in [27]).

[17] As before, we have to assume that each generalized eigenvector of $M^2_V(m^2_{V(\lambda)})$ associated with the eigenvalue $m^2_{V(\lambda)}$ is an ordinary eigenvector.

$$\lim_{q^2\to 0}\left\{M_V^{2}{}''(q^2)\,\xi\right\}, \tag{26}$$

has to exist for each ξ belonging to a basis of the null eigenspace of $M_V^2(0)$. In Sec. 4.1 we will show that the limits (23) and (26) are indeed finite for the photonic eigenvector ξ in the SM at one-loop order.

The discussion of physically meaningful infrared divergences (i.e. the ones that lead to divergent residues) is beyond the scope of this paper, as they change the structure of asymptotic states [37]. In what follows, it will be assumed that an IR regulator has been introduced, if necessary, so that the limits (23) and (26) are finite.

We end this section with an alternative prescription for finding the directions of eigenvectors $\zeta_{V[\lambda_r]}$ corresponding to massless spin 1 particles. At the tree level, when

$$M_V^2(q^2)_{\alpha\beta}=\left[m_V^2\right]_{\alpha\beta}\equiv(\mathcal{T}_\alpha\, v_{(0)})^\top(\mathcal{T}_\beta\, v_{(0)})\,, \tag{27}$$

the eigenvectors of the matrix $M_V^2(0)_{\alpha\beta}$ corresponding to its zero eigenvalues are directly related to the unbroken generators of the gauge group [27]

$$m^2_{V\alpha\beta}\,\theta^\beta=0 \qquad \Leftrightarrow \qquad \theta^\beta\,\mathcal{T}_\beta\, v_{(0)}=0\,.$$

This immediately determines the vectors $\zeta_{V[\lambda_r]}$ corresponding to massless gauge bosons at the zeroth order. In the Landau gauge this prescription generalizes to higher orders owing to the antighost identity [22], specific for this gauge, which guarantees that quantum corrections to 1PI correlation functions of ghosts vanish at zero momentum. This fact is particularly useful when applied to the functions representing the corrections to BRST transformations. Because in the Landau gauge the (anti)ghosts are massless to all orders, the function $\Omega(q^2)^\alpha{}_\beta$ which parametrizes the (renormalized) 1PI two-point ghost–antighost function

$$\frac{\delta}{\delta\hat{\omega}^\beta(q)}\frac{\delta}{\delta\hat{\bar{\omega}}_\alpha(p)}\Gamma\bigg|_0=(2\pi)^4\delta^{(4)}(q+p)\left\{-q^2\Omega(q^2)^\alpha{}_\beta\right\}, \tag{28}$$

must (in our conventions) have the form

$$\Omega(q^2)^\alpha{}_\beta=-\delta^\alpha{}_\beta+\mathcal{O}(\hbar)\,. \tag{29}$$

Existence of unbroken gauge symmetries means that there are vectors Θ^β such that[18]

$$\Theta^\beta\,\mathcal{T}_\beta\, v=0\,. \tag{30}$$

From the antighost identity combined with a Slavnov–Taylor identity it then follows (see the discussion below the formula (85) in Sec. 5.1) that

$$\lim_{q^2\to 0}\left\{M_V^2(q^2)_{\beta\alpha}\,\Omega(q^2)^\alpha{}_\gamma\,\Theta^\gamma\right\}=0\,, \tag{31}$$

which means that the vectors $\zeta_{V[\lambda_r]}$ corresponding to massless gauge bosons are up to normalization given by $\zeta_V^\alpha\propto\Omega(0)^\alpha{}_\gamma\,\Theta^\gamma$.

The identity (31) is interesting in its own right, as it immediately shows, for instance, that the photon in the SM remains massless to all orders. It will also play an important role in the analysis of the unphysical asymptotic states in Sec. 5, in particular in showing that they form Kugo–Ojima quartets.

[18] Recall that v is the complete (and renormalized) VEV, as in Eq. (16).

3. Results in a general renormalizable model

In this section we give one-loop expressions for matrices $M_S^2(p^2)$ and $M_V^2(p^2)$, cf. Eqs. (5) and (17), in a general renormalizable gauge field theory model.

3.1. Parametrization of the action

We assume that the gauge group is a direct product of an arbitrary number of compact simple Lie groups and $U(1)$ groups and that the gauge fields are coupled to scalar and fermionic fields forming arbitrary representations of the gauge group (we assume the representation formed by fermions is nonanomalous). We work with real scalars ϕ^j, real vectors A^α_μ and Weyl fermions χ^a_A (together with their complex conjugates $\overline{\chi}^a_{\dot{A}}$). Recall that the fields ϕ^j are assumed to have all vanishing VEV, and are related to "the symmetric phase" field by $\phi^j_{\rm sym} = \phi^j + v^j$. The classical gauge-invariant action I_0^{GI} is the integral of the Lagrangian density (we follow the conventions of [33])

$$\mathcal{L}_0^{GI} = -\frac{1}{4}\delta_{\alpha\beta} F^\alpha{}_{\mu\nu} F^{\beta\mu\nu} + \frac{1}{2}\delta_{ij}(D_\mu\phi)^i(D^\mu\phi)^j - \mathcal{V}(\phi+v) + \mathcal{L}_0^F\,. \tag{32}$$

Lorentz indices are lowered/raised with the aid of the Minkowski metric $\eta_{\mu\nu}$. The potential $\mathcal{V}(\phi_{\rm sym})$ is a fourth order polynomial parametrized below by the following coupling constants and mass parameters:

$$\lambda_{ijkl} = \mathcal{V}^{(4)}_{ijkl}(v_{(0)}), \qquad \rho_{ijk} = \mathcal{V}'''_{ijk}(v_{(0)}), \qquad m^2_{Sij} = \mathcal{V}''_{ij}(v_{(0)}), \tag{33}$$

where $v_{(0)}$, determined by the condition $\mathcal{V}'_i(v_{(0)}) = 0$, is the first term of the expansion (16) of the complete VEV. The covariant derivative of scalars is given by (15), and the explicit form of $F^\alpha{}_{\mu\nu}$ is

$$F^\alpha{}_{\mu\nu} = \partial_\mu A^\alpha_\nu - \partial_\nu A^\alpha_\mu + e^\alpha{}_{\beta\gamma} A^\beta_\mu A^\gamma_\nu\,,$$

with real structure constants $e^\gamma{}_{\alpha\beta}$ which include the gauge couplings and are defined by the relation $[\mathcal{T}_\alpha, \mathcal{T}_\beta] = \mathcal{T}_\gamma\, e^\gamma{}_{\alpha\beta}$).

The fermionic part of the Lagrangian density reads

$$\mathcal{L}_0^F = i\,\delta_{ab}\,\overline{\chi}^a\overline{\sigma}^\mu\partial_\mu\chi^b + i\,\mathfrak{t}_{\alpha ab}\,\overline{\chi}^a\overline{\sigma}^\mu\chi^b A^\alpha_\mu + \\ -\frac{1}{2}\left(\overline{M}_{F\,ab}\,\chi^a\chi^b + \overline{M}^\star_{F\,ab}\,\overline{\chi}^a\overline{\chi}^b\right) - \frac{1}{2}\phi^j\left(Y_{jab}\,\chi^a\chi^b + Y^\star_{jab}\,\overline{\chi}^a\overline{\chi}^b\right), \tag{34}$$

where $SL(2,\mathbb{C})$ indices have been suppressed

$$\chi^a\chi^b \equiv \chi^{a\,A}\,\chi^b_A\,, \qquad \overline{\chi}^a\,\overline{\sigma}^\mu\,\chi^b \equiv \overline{\chi}^a_{\dot{B}}\,\overline{\sigma}^{\mu\,\dot{B}A}\,\chi^b_A\,,$$

etc. Here $\mathfrak{t}_{\alpha ab} = -\mathfrak{t}^\star_{\alpha ba}$ are matrix elements of anti-Hermitian gauge-group generators ($[\mathfrak{t}_\alpha, \mathfrak{t}_\beta] = \mathfrak{t}_\gamma\, e^\gamma{}_{\alpha\beta}$), while $Y_{jab} = Y_{jba}$ are elements of symmetric Yukawa matrices Y_j. The fermionic matrix $\overline{M}_F$ depends on v

$$\overline{M}_{F\,ab} = \mathscr{M}_{ab} + Y_{jab}\,v^j\,. \tag{35}$$

The coefficients $\mathscr{M}_{ab}$, Y_{jab}, etc. are, of course, constrained by the gauge (and global) symmetries.[19]

In calculating diagrams we find it more convenient to work with four-component Majorana spinors ψ^a

$$\psi^a = \begin{bmatrix} \chi^a_A \\ \overline{\chi}^{a\dot{A}} \end{bmatrix}. \tag{36}$$

For this reason solid lines in diagrams displayed below represent Majorana fields (and are, consequently, non-oriented). We therefore rewrite the fermionic part of the Lagrangian in the following form (discarding total derivatives)

$$\begin{aligned}\mathcal{L}^F_0 &= +\frac{1}{2}\,\bar{\psi}^a\Big\{\delta_{ab}\, i\,\gamma^\mu\,\partial_\mu\psi^b - \Big(\overline{M}_{F\,ab}\,P_L + \overline{M}^\star_{F\,ab}\,P_R\Big)\,\psi^b\Big\} + \\ &\quad +\frac{1}{2!}\, i\, A^\alpha_\mu\;\bar{\psi}^a\,\gamma^\mu\left(\mathfrak{f}_{\alpha ab}\,P_L + \mathfrak{f}^\star_{\alpha ab}\,P_R\right)\psi^b + \\ &\quad -\frac{1}{2!}\,\phi^j\,\bar{\psi}^a\Big(Y_{jab}\,P_L + Y^\star_{jab}\,P_R\Big)\,\psi^b\,,\end{aligned} \tag{37}$$

where $P_{L,R}$ are chiral projections and $\bar{\psi} \equiv \psi^\dagger\,\gamma^0 = \psi^\top\,\mathcal{C}$ is the Dirac-conjugate field.

To generate Green's functions of the quantum theory, the classical action I^{GI}_0 is supplemented with a gauge fixing term and with the ghost fields action, what leads to the BRST invariant tree-level action

$$I_0 = I^{GI}_0 + I^{Rest}_0 = \int \mathrm{d}^4x\,(\mathcal{L}^{GI}_0 + \mathcal{L}^{Rest}_0)\,, \tag{38}$$

where $\mathcal{L}^{Rest}_0$ depends on the Nakanishi–Lautrup fields h_β and the ghost and antighost fields ω^α and $\overline{\omega}_\alpha$; in order to control quantum corrections to the gauge transformations one also introduces terms with the external sources (antifields) [40,41,39] K_i, $\bar{K}_a$, K^μ_α and L_α:

$$\mathcal{L}^{Rest}_0 = s\big(-\overline{\omega}_\alpha\,\partial_\mu A^{\alpha\mu}\big) + L_\alpha\,s(\omega^\alpha) + K_i\,s(\phi^i) + \bar{K}_a\,s(\psi^a) + K^\mu_\alpha\,s(A^\alpha_\mu)\,, \tag{39}$$

where the action on fields of the BRST differential s is given by [40,41,39]

$$\begin{aligned} &s(\phi^i) = \omega^\alpha\,[\mathcal{T}_\alpha(\phi+v)]^i\,, && s(\psi^a) = \omega^\alpha([\mathfrak{f}_\alpha]^a{}_b P_L + [\mathfrak{f}^\star_\alpha]^a{}_b P_R)\psi^b,\\ &s(A^\gamma_\mu) = -\partial_\mu\omega^\gamma + e^\gamma{}_{\alpha\beta}\,\omega^\alpha A^\beta_\mu, && s(\omega^\alpha) = \frac{1}{2}\,e^\alpha{}_{\beta\gamma}\,\omega^\beta\omega^\gamma\,,\\ &s(\overline{\omega}_\alpha) = h_\alpha, && s(h_\alpha) = 0. \end{aligned} \tag{40}$$

The "flavor" indices on constant tensors parameterizing the action are raised/lowered with the aid of standard (Kronecker delta) metrics that appear in Eq. (32). In particular, $[\mathfrak{f}_\alpha]^a{}_b \equiv \mathfrak{f}_{\alpha ab}$, etc.

The first term in $\mathcal{L}^{Rest}_0$ represents the gauge-fixing and ghosts Lagrangian in the Landau gauge. Setting $s(K_i) = s(\bar{K}_a) = s(K^\mu_\alpha) = s(L_\alpha) = 0$ makes the action I^{Rest}_0 a BRST-exact functional: $I^{Rest}_0 = s\,W$. The complete action (38) is then BRST-invariant, $s I_0 = 0$, due to the nilpotency $s^2 = 0$.

[19] These constraints imply, in particular, that $\mathscr{M}_{ab} \equiv 0$ in the SM.

All fields and parameters introduced above are understood as renormalized quantities. In other words, the one-loop action has the form

$$I_1 = I_0 - \hbar\, \delta I_1\,,$$

and contains counterterms δI_1; in the $\overline{\mathrm{MS}}$ scheme of dimensional regularization each term in δI_1 is a singular part of an appropriately chosen 1PI one-loop effective vertex. We also note that I_0 itself contains terms with all powers of $\hbar$, as it depends on the complete (but renormalized) VEV v^i (cf. the formula (16)).

3.2. One-loop self-energies

The formulae collected in this section are valid in the Landau gauge, and are renormalized in the $\overline{\mathrm{MS}}$ scheme [42] of dimensional regularization with the anticommuting γ^5 matrix which in non-anomalous theories is consistent at the one-loop order and preserves chiral gauge symmetries.[20] All loop integrals associated with the diagrams listed in this section were checked against the FeynCalc [43] results.

Without loss of generality we assume that the components ϕ^j, A^α_μ and χ^a_A are chosen in such a way that the tree-level mass-squared matrices are diagonal (and nonnegative)

$$\left[m^2_{S\,ij}\right] = \mathrm{diag}(m^2_{S\,1}, \ldots)\,, \qquad \left[m^2_{V\,\alpha\beta}\right] = \mathrm{diag}(m^2_{V\,1}, \ldots)\,,$$

(cf. Eqs. (33) and (27)) and[21]

$$M_F M_F^\star = \mathrm{diag}(m^2_{F\,1}, \ldots)\,, \tag{41}$$

where (cf. Eqs. (35) and (16))

$$M_{F\,ab} = \mathscr{M}_{ab} + Y_{jab}\, v^j_{(0)}\,.$$

In particular, the pole masses $m^2_{S(\ell)}$ and $m^2_{V(\lambda)}$ are $\mathcal{O}(\hbar)$ perturbations of the appropriate tree-level masses $m^2_{S\,j} = m^2_{S\,jj}$ and $m^2_{V\,\alpha} = m^2_{V\,\alpha\alpha}$.

One-loop 1PI diagrams contributing to $M^2_S(p^2)$ and $M^2_V(p^2)$ in the Landau gauge are shown in Figs. 1 and 2, respectively. Finite (minimally subtracted) parts of these contributions are denoted by $-(4\pi)^{-2}\Delta^S(p^2)$ and $+(4\pi)^{-2}\Sigma^V(p^2)$. In addition, the quantum correction $v_{(1)}$ to the VEV in Eq. (16) contributes to both matrices $M^2_S(p^2)$ and $M^2_V(p^2)$; thus ($s \equiv p^2$)

$$M^2_V(s)_{\alpha\beta} = (\mathcal{T}_\alpha\, v_{(0)})^\top (\mathcal{T}_\beta\, v_{(0)}) - \hbar\, v^\top_{(0)} \{\mathcal{T}_\alpha\,,\ \mathcal{T}_\beta\} v_{(1)} + \frac{\hbar}{(4\pi)^2}\, \Sigma^V(s)_{\alpha\beta} + \mathcal{O}(\hbar^2)\,,$$

$$M^2_S(s)_{ij} = \mathcal{V}''_{ij}(v_{(0)} + \hbar\, v_{(1)}) - \frac{\hbar}{(4\pi)^2}\, \Delta^S(s)_{ij} + \mathcal{O}(\hbar^2)\,. \tag{42}$$

Matrices $\Delta^S(p^2)$ and $\Sigma^V(p^2)$ can be expressed in terms of the (minimally subtracted) one-loop functions a^R and b^R_0 in the dimensional regularization (see e.g. [4])

[20] Since we use the dimensional regularization (rather than dimensional reduction), additional finite counterterms have to be adjusted in supersymmetric models to restore supersymmetry. We do not give explicit expressions for them in what follows.

[21] In realistic model there are Dirac particles and it is more convenient to keep M_F non-diagonal, diagonalizing only the product $M_F M_F^\star$.

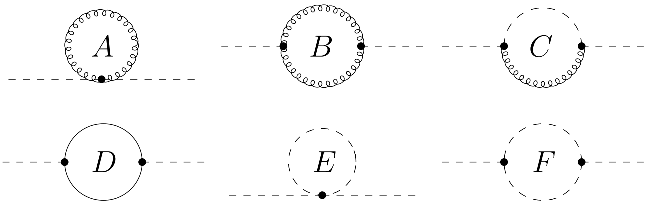

Fig. 1. One-loop contributions to the two-point functions $\widetilde{\Gamma}_{i_1 i_2}(p,-p)$ of scalar fields. Solid lines represent Majorana fermions (36) (see the Lagrangian (37)). At order $\mathcal{O}(\hbar)$ to $\widetilde{\Gamma}_{i_1 i_2}(p,-p)$ contributes also the correction to the VEV, cf. Eqs. (42).

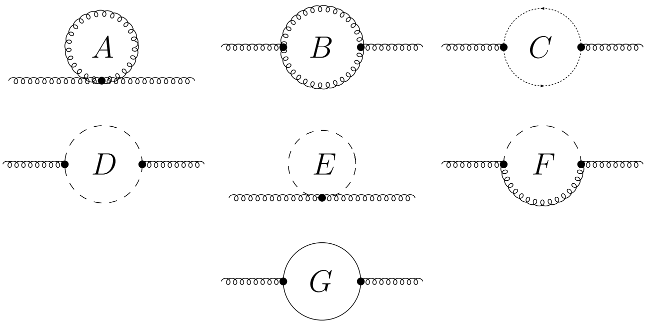

Fig. 2. One-loop contributions to the self-energy $\widetilde{\Gamma}^{\mu\nu}_{\alpha\beta}(p,-p)$ of vector fields. Diagram C represents the ghost–antighost loop. At order $\mathcal{O}(\hbar)$ to $\widetilde{\Gamma}^{\mu\nu}_{\alpha\beta}(p,-p)$ contributes also the correction to the VEV, cf. Eqs. (42).

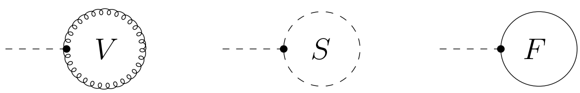

Fig. 3. One-loop contributions to the scalar one-point functions $\widetilde{\Gamma}_i(0)$ necessary to determine quantum-corrected VEV (cf. Eq. (44)).

$$
\begin{aligned}
a^R(m) &= m^2 \left\{ \ln \frac{m^2}{\bar{\mu}^2} - 1 \right\}, \\
b_0^R(p^2, m_1, m_2) &= \int_0^1 \mathrm{d}x \ln \frac{x(x-1)p^2 + (1-x)m_1^2 + x\, m_2^2 - i\,0}{\bar{\mu}^2},
\end{aligned}
\tag{43}
$$

where $\bar{\mu}$ is the renormalization scale of the $\overline{\mathrm{MS}}$ scheme, related to the usual 't Hoot mass unit μ_H via $\bar{\mu} \equiv \mu_H \sqrt{4\pi}\, e^{-\gamma_E/2}$.

We begin with one-loop corrections to scalar tadpoles which are shown in Fig. 3. They yield the following equation for $v_{(1)}$

$$
\begin{aligned}
0 = &-\mathcal{V}_i'(v_{(0)} + \hbar\, v_{(1)}) + \frac{\hbar}{(4\pi)^2} \Big\{ 3 \sum_{\alpha j} [\mathcal{T}_\alpha^2]_{ij}\, v_{(0)}^j \left[a^R(m_{V\alpha}) + \frac{2}{3} m_{V\alpha}^2 \right] + \\
&- \frac{1}{2} \sum_j \rho_{ijj}\, a^R(m_{Sj}) + \sum_{bc} (M_{Fbc} Y_{icb}^\star + M_{Fbc}^\star Y_{icb}) a^R(m_{Fb}) \Big\} + \mathcal{O}(\hbar^2).
\end{aligned}
\tag{44}
$$

The contribution $\Delta^S(p^2)_{ij}$ to the scalar two-point function reads ($s \equiv p^2$)

$$\Delta^S(s)_{ij} = 3\sum_\alpha [\mathcal{T}_\alpha^2]_{ij}\left[a^R(m_{V\alpha}) + \frac{2}{3}m_{V\alpha}^2\right] - 4\sum_{\alpha k}\mathcal{T}_{\alpha ik}\,\mathcal{T}_{\alpha kj}\,S_C(s, m_{V\alpha}, m_{Sk}) +$$
$$-\frac{1}{2}\sum_{\alpha\beta}[\{\mathcal{T}_\alpha\,,\ \mathcal{T}_\beta\}v_{(0)}]_i\,[\{\mathcal{T}_\alpha\,,\ \mathcal{T}_\beta\}v_{(0)}]_j\,S_B(s, m_{V\alpha}, m_{V\beta}) +$$
$$+\sum_{abcd}\Big\{(Y_{iab}\,\delta_{bc}\,Y^\star_{jcd}\,\delta_{da} + \text{cc.})\,S_D(s, m_{Fb}, m_{Fd}) +$$
$$+\,(Y_{iab}\,M^\star_{Fbc}\,Y_{jcd}\,M^\star_{Fda} + \text{cc.})\,b_0^R(s, m_{Fb}, m_{Fd})\Big\} +$$
$$-\frac{1}{2}\sum_k \lambda_{ijkk}\,a^R(m_{Sk}) - \frac{1}{2}\sum_{kn}\rho_{kin}\,\rho_{njk}\,b_0^R(s, m_{Sn}, m_{Sk})\,, \tag{45}$$

where cc. indicates the complex conjugation of the preceding term. The following combinations of basic one-loop functions have been introduced

$$S_C(s, m_V, m_S) = \frac{1}{4}\Bigg\{\,a^R(m_V) - a^R(m_S) + \left(s - m_S^2\right)\frac{a^R(m_V)}{m_V^2} +$$
$$+\left(2s + 2m_S^2 - m_V^2\right)b_0^R(s, m_V, m_S) +$$
$$-\frac{\left(s - m_S^2\right)^2}{m_V^2}\left[b_0^R(s, m_V, m_S) - b_0^R(s, 0, m_S)\right]\Bigg\}\,, \tag{46}$$

$$S_B(s, m_1, m_2) = 2 + \frac{1}{4\,m_1^2\,m_2^2}\Big\{m_2^2\,a^R(m_1) + m_1^2\,a^R(m_2) + s^2\,b_0^R(s, 0, 0) +$$
$$-(m_2^2 - s)^2\,b_0^R(s, 0, m_2) - (m_1^2 - s)^2\,b_0^R(s, m_1, 0) +$$
$$+\big[\,(m_2^2 - s)(m_1^2 - s) + m_1^2(m_1^2 - s) + m_2^2(m_2^2 - s) +$$
$$+9\,m_1^2\,m_2^2\,\big]\,b_0^R(s, m_1, m_2)\Big\}\,, \tag{47}$$

and

$$S_D(s, m_1, m_2) = a^R(m_2) + \left\{m_1^2 - \frac{s}{2}\right\}\,b_0^R(s, m_1, m_2)\,. \tag{48}$$

The reality conditions (11) are violated whenever b_0^R has a non-vanishing imaginary part.

Contributions of massless vectors is obtained by taking in the formulae (46)–(47) the limits $m_V \to 0$. We also note that $S_C(0, m_V, m_S) = 0$.

The contribution $\Sigma^V(s)_{\alpha\beta}$ to the two-point function of vector fields reads

$$\Sigma^V(s)_{\alpha\beta} = \sum_{\epsilon\gamma} e^\epsilon{}_{\alpha\gamma}\,e^\gamma{}_{\beta\epsilon}\,V_{ABC}(s, m_{V\epsilon}, m_{V\gamma}) + \sum_{ij}\mathcal{T}_{\alpha ij}\,\mathcal{T}_{\beta ji}\,V_{DE}(s, m_{Si}, m_{Sj}) +$$

$$+\sum_{\gamma i}[\{\mathcal{T}_\gamma\,,\ \mathcal{T}_\alpha\}v_{(0)}]_i\,[\{\mathcal{T}_\gamma\,,\ \mathcal{T}_\beta\}v_{(0)}]_i\,V_F(s,m_{V\gamma},m_{Si})+$$

$$+\sum_{abcd}\Big\{(\mathfrak{f}_{\alpha ab}\,\delta_{bc}\,\mathfrak{f}_{\beta cd}\,\delta_{da}+\text{cc.})\,V_G(s,m_{Fb},m_{Fd})+$$

$$-(\mathfrak{f}_{\alpha ab}\,M^\star_{Fbc}\,\mathfrak{f}^\star_{\beta cd}\,M_{Fda}+\text{cc.})\,b_0^R(s,m_{Fb},m_{Fd})\Big\}. \tag{49}$$

The functions V are defined in terms of the auxiliary function

$$A(s,m_1,m_2)=\frac{m_1^2-m_2^2}{12s}\left[a^R(m_1)-a^R(m_2)-\left(m_1^2-m_2^2\right)b_0^R(s,m_1,m_2)\right]+$$

$$+\frac{1}{12}\left[2m_1^2+2m_2^2-s\right]b_0^R(s,m_1,m_2)+$$

$$+\frac{1}{12}\left[a^R(m_1)+a^R(m_2)\right]+\frac{s}{18}-\frac{1}{6}\left(m_1^2+m_2^2\right), \tag{50}$$

and read

$$V_{ABC}(s,m_1,m_2)=\frac{5s}{3}-\left\{a^R(m_1)+a^R(m_2)\right\}+$$

$$+\frac{1}{2m_1^2m_2^2}\Big\{\left[m_1^4+10\,m_1^2m_2^2+m_2^4+10\left(m_1^2+m_2^2\right)s+s^2\right]A(s,m_1,m_2)+$$

$$-\left[m_1^4+10\,m_1^2s+s^2\right]A(s,m_1,0)+$$

$$-\left[m_2^4+10\,m_2^2s+s^2\right]A(s,0,m_2)+$$

$$+\left[s^2-2\,m_1^2m_2^2\right]A(s,0,0)\Big\}, \tag{51}$$

$$V_{DE}(s,m_1,m_2)=2\,A(s,m_1,m_2)-\frac{1}{2}a^R(m_1)-\frac{1}{2}a^R(m_2)\,, \tag{52}$$

$$V_F(s,m_V,m_S)=b_0^R(s,m_V,m_S)-\frac{1}{m_V^2}\left\{A(s,m_V,m_S)-A(s,0,m_S)\right\}, \tag{53}$$

and

$$V_G(s,m_1,m_2)=\frac{1}{2}\Big\{a^R(m_1)+a^R(m_2)+\left(m_1^2+m_2^2-s\right)b_0^R(s,m_1,m_2)+$$

$$-4A(s,m_1,m_2)\Big\}. \tag{54}$$

Again, in contributions of massless gauge bosons the limit $m_V\to 0$ is understood and, again, the imaginary part of b_0^R violates the reality of $M_V^2(s)$.

It will be useful to have also the one-loop correction to the ghost–antighost self-energy which at one-loop is given by the single diagram of Fig. 4. It gives the following factor $\Omega(q^2)^\alpha_{\ \gamma}$ in the two-point function (28)

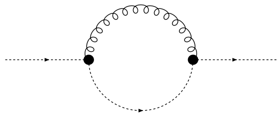

Fig. 4. One-loop contribution to the ghost–antighost self-energy.

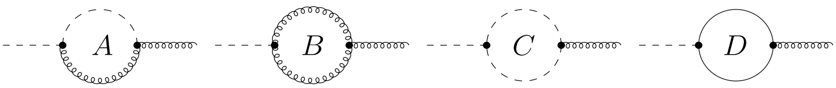

Fig. 5. One-loop contributions to $\widetilde{\Gamma}^{\nu}_{i\beta}(p,-p)$. At order $\mathcal{O}(\hbar)$ to $\widetilde{\Gamma}^{\nu}_{i\beta}(p,-p)$ contributes also the correction to the VEV (cf. Eq. (57)).

$$\Omega(q^2)^{\alpha}{}_{\gamma} = -\delta^{\alpha}{}_{\gamma} + \frac{\hbar}{(4\pi)^2}\sum_{\beta\epsilon} e^{\alpha}{}_{\beta\epsilon}\, e^{\epsilon}{}_{\beta\gamma}\, \mathcal{H}(q^2, m_{V\beta}) + \mathcal{O}(\hbar^2)\,, \tag{55}$$

where

$$\mathcal{H}(s,m) = \frac{1}{2}\, b_0^R(s,m,0) - \frac{1}{4s}\left\{m^2\, b_0^R(s,m,0) - a^R(m)\right\} + \\ -\frac{1}{4m^2}\left\{s\left[b_0^R(s,m,0) - b_0^R(s,0,0)\right] - a^R(m)\right\}. \tag{56}$$

Notice that because $a^R(m) = m^2\, b_0^R(0,m,0)$, the function $\mathcal{H}(s,m)$ does not have a pole at $s=0$.

Finally, 1PI diagrams contributing to the scalar-vector two-point function at one-loop are shown in Fig. 5. There is also an additional contribution originating from a correction to the VEV; schematically we can write

$$\widetilde{\Gamma}^{\nu}_{j\beta}(p,-p) = -i\, p^{\nu}\, \mathcal{T}_{\beta jk}(v^k_{(0)} + \hbar\, v^k_{(1)}) + [\text{Fig. 5}] + \mathcal{O}(\hbar^2)\,. \tag{57}$$

We do not need the expression for $\widetilde{\Gamma}^{\nu}_{j\beta}(p,-p)$ (just as we do not need the expression for the matrix $\mathscr{L}(q^2)$ in Eq. (17)). Nevertheless, the fermionic contribution to this function (i.e. the finite part of diagram D in Fig. 5) will turn out to be useful in Sec. 4.2

$$\widetilde{\Gamma}^{\nu}_{j\beta}(p,-p)_{[5.D]} = \hbar\, \frac{2\,i\, p^{\nu}}{(4\pi)^2}\sum_{abcd}\left\{(Y_{jab}\, M^{\star}_{Fbc}\, \mathfrak{f}^{\star}_{\beta cd}\, \delta_{da} + \text{cc.})\, J(p^2, m_{Fd}, m_{Fb})\right\}, \tag{58}$$

where

$$J(s,m_1,m_2) = \frac{1}{2s}\left\{a^R(m_1) - a^R(m_2) + \left[m_2^2 - m_1^2 - s\right] b_0^R(s,m_1,m_2)\right\}.$$

For future reference, we note that contributions of fermionic loops of Figs. 1.D, 5.D and 3.F are related as follows

$$0 = -i\, p_{\mu}\, \widetilde{\Gamma}^{\mu}_{j\alpha}(p,-p)_{[5.D]} + (\mathcal{T}_{\alpha}\, v_{(0)})^k\, \widetilde{\Gamma}_{jk}(p,-p)_{[1.D]} + [\mathcal{T}_{\alpha}]^k{}_j\, \widetilde{\Gamma}_k(0)_{[3.F]}\,, \tag{59}$$

where $(4\pi)^2\, \widetilde{\Gamma}_{ij}(p,-p)_{[1.D]}$ is given by the fourth term in Eq. (45) (with $s \equiv p^2$), while $\widetilde{\Gamma}_i(0)_{[3.F]}$ represents the term with $a(m_F)$ on the right-hand-side of Eq. (44).

It is perhaps worth stressing, for completeness, that the matrix $\mathscr{L}(q^2)$ in Eq. (17) as well as the two-point function (57) are (at one-loop order) entirely fixed in terms of the matrices $M_V^2(p^2)$, $M_S^2(p^2)$ and (55) by the gauge-symmetry (see a discussion below Eq. (88) in Sec. 5.1). Thus, the above results give the complete set of bosonic two-point functions in the Landau gauge.

4. Examples

4.1. Corrections to electroweak mixing

The matrix $M_V^2(p^2)$ parameterizing the two-point function (17) of the SM vector fields is block-diagonal. It has two 2×2 blocks corresponding to the pairs (Z_μ, A_μ), (W^1_μ, W^2_μ) and one 8×8 block corresponding to gluons; in the last two blocks the matrix $M_V^2(p^2)$ is proportional to the identity matrix (see e.g. [1]). Here Z_μ and A_μ denote, as usually, the eigenfields of the tree-level mass-squared matrix with eigenvalues m_Z^2 and 0. The generic formulae of Sec. 3 yield the following one-loop expression for the (Z, A) block of $M_V^2(0)$ in the $\overline{\rm MS}$ scheme

$$M_V^2(0) = \begin{bmatrix} m_Z^2 + \hbar\, \mathcal{a} & \hbar\, \mathcal{b} \\ \hbar\, \mathcal{b} & 0 \end{bmatrix} + \mathcal{O}(\hbar^2)\,, \tag{60}$$

with

$$\begin{aligned} \mathcal{a} = \frac{1}{(4\pi)^2 v_{H(0)}^2}\Bigg\{ &\frac{6m_H^2 m_Z^4}{m_H^2 - m_Z^2}\left[\ln\left(\frac{m_H}{\bar\mu}\right) - \frac{5}{12}\right] - \frac{6m_Z^6}{m_H^2 - m_Z^2}\left[\ln\left(\frac{m_Z}{\bar\mu}\right) - \frac{5}{12}\right] + \\ &+\frac{1}{2}m_H^2 m_Z^2 - \left(24 m_W^4 - 12 m_W^2 m_Z^2\right)\left[\ln\left(\frac{m_W}{\bar\mu}\right) - \frac{5}{12}\right] + \\ &-12 m_Z^2 \sum_{quarks} m_q^2 \ln\left(\frac{m_q}{\bar\mu}\right) - 4m_Z^2 \sum_{\ell = e\mu\tau} m_\ell^2 \ln\left(\frac{m_\ell}{\bar\mu}\right)\Bigg\} + 2m_Z^2 \frac{v_{H(1)}}{v_{H(0)}}\,, \end{aligned} \tag{61}$$

where m_X is the tree-level mass of the particle X and $v_{H(1)}$ in the last term denotes the corrections to the tree-level VEV $v_{H(0)}$ of the (symmetric phase) Higgs doublet field

$$\mathcal{H} \equiv \mathcal{H}_{\rm sym} = \frac{1}{\sqrt{2}}\begin{pmatrix} G_1 + i\,G_2 \\ H + i\,G_Z \end{pmatrix} + \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v_{H(0)} + \hbar\, v_{H(1)} + \mathcal{O}(\hbar^2) \end{pmatrix}. \tag{62}$$

The formula (44) for the one-loop correction to the VEV yields here

$$\begin{aligned} v_{H(1)} = \frac{2}{(4\pi)^2 m_H^2\, v_{H(0)}}\Bigg\{ &6\sum_{quarks} m_q^2\, a^R(m_q) + 2\sum_{\ell = e\mu\tau} m_\ell^2\, a^R(m_\ell) - \frac{3}{4}m_H^2\, a^R(m_H) + \\ &-3m_W^2\left[a^R(m_W) + \frac{2}{3}m_W^2\right] - \frac{3}{2}m_Z^2\left[a^R(m_Z) + \frac{2}{3}m_Z^2\right]\Bigg\}. \end{aligned} \tag{63}$$

At one-loop the factor $\mathcal{b}$ in the off-diagonal element of (60) reads

$$\mathcal{b} = -\frac{3\,e}{8\pi^2}\frac{m_Z}{v_{H(0)}} m_W^2\left[\ln\left(\frac{m_W}{\bar\mu}\right) - \frac{5}{12}\right],$$

where

$$e = 2\frac{m_W}{v_{H(0)}}\sqrt{1 - \frac{m_W^2}{m_Z^2}}\,,$$

is the renormalized charge coupling constant.

We see that the vector

$$\zeta_{[\text{photon}]} = \mathcal{N} \begin{bmatrix} -\hbar\, \mathfrak{b}/m_Z^2 \\ 1 \end{bmatrix} + \mathcal{O}(\hbar^2)\,,$$

corresponds to the zero eigenvalue of the matrix (60):

$$M_V^2(0)\, \zeta_{[\text{photon}]} = \mathcal{O}(\hbar^2)\,, \tag{64}$$

which means that the photon is massless to one-loop accuracy. The normalization factor $\mathcal{N}$ can be obtained from Eq. (25). To this end one needs the derivative of the matrix $M_V^2(q^2)$. Its ZZ element in the limit $q^2 \to 0$ is singular:

$$M_V^{2\,\prime}(q^2)_{ZZ} = \frac{\hbar\, m_Z^2}{48\pi^2 v_{H(0)}^2} \left\{(6-1)\ln(-q^2/\bar{\mu}^2) + \mathcal{O}((q^2)^0)\right\} + \mathcal{O}(\hbar^2)\,. \tag{65}$$

The factor of 6 in the bracket originates from contributions of neutrinos, while -1 is the contribution of unphysical massless gauge degrees of freedom. The AA and AZ elements of this derivative are regular and read

$$M_V^{2\,\prime}(0)_{AA} = \frac{\hbar\, e^2}{6\pi^2} \left\{ \sum_{\ell=e\mu\tau} \ln(m_\ell/\bar{\mu}) + 3 \sum_{quarks} Q_q^2 \ln(m_q/\bar{\mu}) - 3\ln(m_W/\bar{\mu}) - \frac{11}{16} \right\} + \\ + \mathcal{O}(\hbar^2)\,, \tag{66}$$

with $Q_q = +2/3, -1/3$ denoting the electric charge of quark q, and

$$M_V^{2\,\prime}(0)_{ZA} = \frac{\hbar\, e}{(4\pi)^2} \frac{1}{18 m_Z v_{H(0)}} \Big\{ 24\left(4m_W^2 - 3m_Z^2\right) \sum_{\ell=e\mu\tau} \ln(m_\ell/\bar{\mu}) + \\ +16\left(8m_W^2 - 5m_Z^2\right) \sum_{up-quarks} \ln\left(m_q/\bar{\mu}\right) + 8\left(4m_W^2 - m_Z^2\right) \sum_{down-quarks} \ln\left(m_q/\bar{\mu}\right) + \\ -12\left(24m_W^2 + m_Z^2\right) \ln(m_W/\bar{\mu}) - 66m_W^2 + 41m_Z^2 \Big\} + \mathcal{O}(\hbar^2)\,. \tag{67}$$

We thus see that, despite the singular behavior of $M_V^{2\,\prime}(q^2)_{ZZ}$, the product $M_V^{2\,\prime}(q^2)\, \zeta_{[\text{photon}]}$ is finite in the limit $q^2 \to 0$ to one-loop accuracy, in agreement with conditions formulated in Sec. 2.2. We have checked that $M_V^{2\,\prime\prime}(q^2)\, \zeta_{[\text{photon}]}$ is also finite for $q^2 \to 0$, and therefore the propagator of vector fields has in the (Z, A) block a pole at $q^2 = 0$. In particular, the correctly normalized eigenvector $\zeta_{[\text{photon}]}$ has the form

$$\zeta_{[\text{photon}]} = (1 + \frac{1}{2} M_V^{2\,\prime}(0)_{AA}) \begin{bmatrix} -\hbar\, \mathfrak{b}/m_Z^2 \\ 1 \end{bmatrix} + \mathcal{O}(\hbar^2) = \begin{bmatrix} -\hbar\, \mathfrak{b}/m_Z^2 \\ 1 + \frac{1}{2} M_V^{2\,\prime}(0)_{AA} \end{bmatrix} + \mathcal{O}(\hbar^2). \tag{68}$$

Using Eq. (21) we get the decomposition of the asymptotic fields $\mathbf{Z}_\mu$ and $\mathbf{A}_\mu$ corresponding to Z_μ and A_μ

$$\mathbf{Z}_\mu = -\frac{\hbar\, \mathfrak{b}}{m_Z^2} \mathbb{A}_\mu + \mathcal{O}(\hbar^2) + \dots\,, \\ \mathbf{A}_\mu = \left\{1 + \frac{1}{2} M_V^{2\,\prime}(0)_{AA}\right\} \mathbb{A}_\mu + \mathcal{O}(\hbar^2) + \dots\,, \tag{69}$$

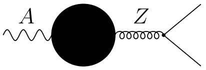

Fig. 6. Diagrams with the external line corrections that reproduce the operator in Eq. (70).

where $\mathbb{A}_\mu$ is a canonically normalized free massless vector field in the Coulomb gauge. The ellipsis indicates the contributions of unphysical modes discussed in Section 5.[22]

Because $\text{б} \neq 0$, the amputated correlation functions of the Z_μ field contribute to transition amplitudes with photons. Taking, for instance, the coupling between Z_μ and fermions (cf. Eq. (37))

$$\mathcal{L}_0^F \supset \frac{1}{2}\, i\, Z_\mu\, \bar{\psi}^a \gamma^\mu \left(\mathfrak{f}_{Zab}\, P_L + \mathfrak{f}^\star_{Zab}\, P_R\right) \psi^b\,,$$

we see that б gives the following contribution to the S-operator

$$S_{\mathrm{mix}} = \frac{\hbar\, \text{б}}{2m_Z^2} \int \mathrm{d}^4x\, \mathbb{A}_\mu\, \bar{\boldsymbol{\psi}}^{a_1} \gamma^\mu \left(\mathfrak{f}_{Za_1a_2}\, P_L + \mathfrak{f}^\star_{Za_1a_2}\, P_R\right) \boldsymbol{\psi}^{a_2}\,. \tag{70}$$

$\boldsymbol{\psi}^a$ are here the asymptotic fields corresponding to ψ^a. In certain extensions of the SM this term contributes to e.g. decays of heavy neutrinos into light ones and photons. The S_{mix} term is by no means surprising; it can be recovered by ignoring the LSZ formalism and including instead the terms of the Dyson series corresponding to diagrams shown in Fig. 6. By contrast, in the proper LSZ approach which we have extended here to the case of fields subject to mixing, the amplitudes are inferred directly from the amputated correlation functions. With our prescription, one can find the external line factors $\zeta^\alpha_{V[\lambda_r]}$ which are correctly normalized also at higher orders, what is essential for unitarity.[23]

We can also use the example of the Z-photon mixing to demonstrate how the relation (31) determines the direction (but not the normalization) of the eigenvector $\zeta_{[\mathrm{photon}]}$. The advantage of this prescription lies in the small number of diagrams contributing to the ghost–antighost self-energy; at one-loop in a general renormalizable gauge theory there is only one diagram (shown in Fig. 4) contributing to the ghost–antighost self-energy, while seven diagrams (those of Fig. 2) can contribute to the self-energy of vector bosons. In the SM, the matrix $\Omega(q^2)$ appearing in the 1PI two-point function (28) has the same block structure as the matrix $M_V^2(q^2)$ discussed above. We are interested in its $(Z,\ A)$ block, which has the form (cf. Eq. (55)):

$$\Omega(0) = -\mathbb{1} - \frac{\hbar}{8\pi^2}\mathcal{H}(0, m_W) \begin{bmatrix} \dfrac{4m_W^4}{m_Z^2\, v_{H(0)}^2} & \dfrac{2\,e\,m_W^2}{m_Z\, v_{H(0)}} \\ \dfrac{2\,e\,m_W^2}{m_Z\, v_{H(0)}} & e^2 \end{bmatrix} + \mathcal{O}(\hbar^2)\,, \tag{71}$$

with

[22] We stress that, while rigorously only eigenvectors $\zeta_{V[\lambda_r]}$ corresponding to stable particles enter the decomposition (21), the factorization (20) of pole residues is correct for complex poles as well. In particular, a (complex) eigenvector $\zeta_{[Z]}$ associated with the Z boson can be useful in the study of properties of the resonance [9–12]. If, however, the Z boson is treated as a stable particle, then the corresponding free vector field $\mathbb{Z}_\mu$ (in the unitarity gauge) should be also included in Eqs. (69). Its "content" in the asymptotic fields $\mathbf{Z}_\mu$ and $\mathbf{A}_\mu$ is then determined by the eigenvector $\zeta_{[Z]} \approx \mathrm{Re}(\zeta_{[Z]})$ associated with the Z pole.

[23] We also note that while the term $q_\mu q_\nu/q^2$ in the numerator of the propagator of the Z field, makes the diagram of Fig. 6 somewhat singular in the Landau gauge, the determination of $\zeta^\alpha_{[\mathrm{photon}]}$ factor is completely free from singularities.

$$\mathcal{H}(0, m_W) = \frac{1}{8}\{12 \ln(m_W/\bar{\mu}) - 5\},$$

(the correction to $\Omega(0)$ has vanishing determinant, which reflects the fact that ghost of the Abelian ideal $U(1)_Y$ are noninteracting). In the SM case, the quantum-corrected VEV v has the same direction as does the tree-level one $v_{(0)}$, and therefore the generator $\mathcal{T}_\alpha = \mathcal{T}_A$, to which the A_μ field couples at the tree-level, remains unbroken also at one-loop order. Thus, the vector Θ that fulfills the condition (30) can be chosen as (in the (Z, A) subspace)

$$\Theta = \begin{bmatrix} 0 \\ -1 \end{bmatrix},$$

so that

$$\Omega(0)\Theta = \left\{1 + e^2 \frac{\hbar}{8\pi^2}\mathcal{H}(0, m_W)\right\} \begin{bmatrix} \frac{\hbar}{4\pi^2} \frac{e\, m_W^2}{m_Z\, v_{H(0)}} \mathcal{H}(0, m_W) \\ 1 \end{bmatrix} + \mathcal{O}(\hbar^2). \tag{72}$$

This is, up to a proportionality factor, the photon eigenvector (68), as expected.

In order to illustrate the role of the limit in Eq. (31), we give here results in the gluonic block, where $M_V^2(q^2) \propto \mathbb{1}$ and $\Omega(q^2) \propto \mathbb{1}$ with the following proportionality factors

$$M_V^2(p^2) = \frac{\hbar\, g_s^2\, p^2}{12\pi^2}\left\{\frac{1}{16}\left[97 - 78\, \ln(-p^2/\bar{\mu}^2)\right] + \sum_{quarks} \ln(m_q/\bar{\mu}) + \mathcal{O}(p^2)\right\} + \mathcal{O}(\hbar^2),$$

and

$$\Omega(p^2) = -1 - \frac{3\,\hbar\, g_s^2}{64\pi^2}\{3\, \ln(-p^2/\bar{\mu}^2) - 4\} + \mathcal{O}(\hbar^2).$$

Hence, Eq. (31) holds also for vectors Θ pointing in the directions of $SU(3)_C$ generators, at least in the perturbative regime.

4.2. Singlet Majoron model

In order to illustrate the usefulness of the condition (22), which determines the vectors $\zeta_{S[\ell_r]}$ associated with physical massless spin 0 particles, we study in this section the singlet Majoron model [34]. Its additional (with respect to the SM – see e.g. [1]) fermionic fields are made up of three gauge-sterile Weyl fields ("neutrino singlets") N_A^i, $i = 1, 2, 3$, and their complex conjugates $\overline{N}_{\dot{A}}^i$. The scalar sector of the model consists of the usual electroweak scalar doublet (62), and a new gauge-sterile complex scalar $\varphi \equiv \varphi_{\rm sym}$ which carries two units of the lepton number. This field couples only to the sterile neutrinos and to the electroweak doublet $\mathscr{H}$; the scalar potential consistent with gauge symmetries and the lepton number symmetry reads

$$\mathcal{V}(\mathscr{H}, \varphi) = -m_1^2\, \mathscr{H}^\dagger \mathscr{H} - m_2^2\, \varphi^\star \varphi + \lambda_1\, (\mathscr{H}^\dagger \mathscr{H})^2 + 2\lambda_3\, \mathscr{H}^\dagger \mathscr{H}\, \varphi^\star \varphi + \lambda_2\, (\varphi^\star \varphi)^2.$$

The Yukawa couplings of the model are given by

$$\mathcal{L}_{\rm Y} = \mathcal{L}_{\rm Y}^{\rm SM} + \left\{Y_{ji}^{\nu} N^{jA} \mathscr{H}^\top \epsilon L_A^i - \frac{1}{2} Y_{ji}^{M} \varphi N^{jA} N_A^i\right\} + \text{H.c.}.$$

$\mathcal{L}_{\rm Y}^{\rm SM}$ represents here the Yukawa couplings of the SM [1], L_A^i are lepton $SU(2)_L$ doublets

$$L^i_A \equiv \begin{pmatrix} \nu^i_A \\ e^i_A \end{pmatrix},$$

with the index $i = 1, 2, 3$ labeling the three families and ϵ is the antisymmetric $SU(2)_L$ metric.

We are interested here in the phase in which both symmetries: the electroweak one and lepton number one, are spontaneously broken. Exploiting the symmetries of the action we can assume that $\langle\varphi\rangle$ is real

$$\varphi \equiv \varphi_{\rm sym} = \frac{1}{\sqrt{2}}(S + i\,G_\varphi) + \frac{1}{\sqrt{2}}(v_{\varphi(0)} + \hbar\, v_{\varphi(1)} + \mathcal{O}(\hbar^2))\,.$$

The parametrization of $\mathscr{H}$ is given by (62). The tree-level VEVs are related to the mass parameters of the potential by

$$m_1^2 = \lambda_3\, v^2_{\varphi(0)} + \lambda_1\, v^2_{H(0)}\,, \qquad\qquad m_2^2 = \lambda_3\, v^2_{H(0)} + \lambda_2\, v^2_{\varphi(0)}\,.$$

Linear combinations $(h, \underline{h})$ of the fields (S, H) are then eigenstates of the tree-level mass-squared matrix, with the eigenvalues $m^2_{\rm I}$ and $m^2_{\rm II}$; all other scalars are massless (would-be) Goldstone bosons.

By a unitary rotation of the three sterile neutrinos N^i the Yukawa matrix Y^M can be brought into a diagonal and non-negative form. The matrix Y^ν is then, in general, non-diagonal. However, as the sole purpose of this section is to illustrate the use of the condition (22), we will simply assume that also Y^ν is positive and diagonal so that both matrices, Y^M and Y^ν can be unambiguously expressed in terms of the masses of the physical light and heavy neutrinos, denoted (with a little abuse of notation) by m_{ν_i} and m_{N_i}. At the one-loop order, the matrix $M^2_S(p^2)$ obtained using the formulae (42) and (45) is then block diagonal with the blocks corresponding to pairs[24] (G_1, G_2), (G_Z, G_φ), and $(h, \underline{h})$. For vanishing p^2, the first two blocks of the matrix $M^2_S(p^2)$ vanish in agreement with the Goldstone theorem; since these results from nontrivial cancellations between the contributions to the one-loop 1PI self-energies and the one-loop corrections to the VEVs, the explicit expressions for the VEVs (obtained from the general formula (44)) are given in Appendix A (Eqs. (163)–(164)) for completeness.

We are interested in the block of $M^2_S(p^2)$ corresponding to the neutral (would-be) Goldstone bosons (G_Z, G_φ). The matrix that appears in the normalization condition (9) has, after reduction to this block, the following form

$$\mathbb{1} - M^{2\,\prime}_S(0) = \begin{bmatrix} 1 + \hbar\,\mathbf{a} & \hbar\,\mathbf{b} \\ \hbar\,\mathbf{b} & 1 + \hbar\,\mathbf{c} \end{bmatrix} + \mathcal{O}(\hbar^2)\,. \tag{73}$$

The formulae for **a**, **b** and **c** follow from Eqs. (42) and (45), and are given in Appendix A, Eqs. (165)–(167).

The null eigenvector $\zeta_{S[\rm Maj]}$ corresponding to the physical Goldstone boson (Majoron) has to obey the condition (22). The gauge symmetry generators relevant to our problem are $\mathcal{T}_\alpha = \mathcal{T}_Z, \mathcal{T}_A$ to which the Z_μ and A_μ fields couple at the tree-level. The latter is unbroken, $\mathcal{T}_A v = 0$, while $\mathcal{T}_Z v$, in the $(G_1, G_2, G_Z, G_\varphi, h, \underline{h})$ coordinates, reads

$$\mathcal{T}_Z\, v = (m_Z\, v_H / v_{H(0)})\ [0,\ 0,\ 1,\ 0,\ 0,\ 0]^\top\,,$$

(m_Z is the tree-level mass of the Z boson). Thus, taking into account the normalization condition (9) we get

[24] In the first block $M^2_S(p^2)$ is proportional to the identity matrix.

Fig. 7. External line corrections reproducing the effects of the operator in Eq. (75). Only fermions (more specifically, neutrinos) contribute to the mixed self-energies.

$$\zeta_{S[\mathrm{Maj}]} = [0,\ 0,\ -\hbar\,\mathbf{b},\ 1-\hbar\,\mathbf{c}/2,\ 0,\ 0]^\top + \mathcal{O}(\hbar^2)\,.$$

The correctly normalized eigenvector associated with the unphysical neutral would-be Goldstone boson has the form

$$\zeta_{S[\mathrm{unph}]} = [0,\ 0,\ 1-\hbar\,\mathbf{a}/2,\ 0,\ 0,\ 0]^\top + \mathcal{O}(\hbar^2)\,.$$

The formula (12) gives, therefore, the following decomposition of asymptotic fields $\mathbf{G}_Z$ and $\mathbf{G}_\varphi$ corresponding to G_Z and G_φ

$$\begin{aligned}
\mathbf{G}_Z &= (1-\hbar\,\mathbf{a}/2)\mathbb{G}_Z - \hbar\,\mathbf{b}\,\mathbb{G}_\varphi + \mathcal{O}(\hbar^2)\,,\\
\mathbf{G}_\varphi &= (1-\hbar\,\mathbf{c}/2)\mathbb{G}_\varphi + \mathcal{O}(\hbar^2)\,,
\end{aligned} \tag{74}$$

where $\mathbb{G}_\varphi$ ($\mathbb{G}_Z$) is the canonically normalized free scalar field constructed out of the operators creating/annihilating states of the physical (unphysical) massless spin 0 particles. In particular, the amputated correlation functions of G_Z contribute to transition amplitudes of the Majoron. By contrast, the amputated correlation functions of the Z_μ field in the Landau gauge cannot contribute to transition amplitudes of (physical) scalar particles.

From the Lagrangian (37) one can read off that $\mathbf{b} \neq 0$ gives rise to the following term in the S-operator

$$S_{\mathrm{mix}} = \hbar\,\mathbf{b}\,\frac{i}{2!}\int \mathrm{d}^4x\,\mathbb{G}_\varphi\,\bar{\boldsymbol{\psi}}^{a_1}\left\{Y_{za_1a_2}\,P_L + Y^\star_{za_1a_2}\,P_R\right\}\boldsymbol{\psi}^{a_2}\,, \tag{75}$$

where the index z on the Yukawa matrices corresponds to the $\phi^z \equiv G_Z$ component of the scalar field, and $\boldsymbol{\psi}^a$ are the asymptotic fields associated with the interpolating fields ψ^a. This one-loop result is consistent with the one obtained by ignoring the LSZ formalism and including the terms of the Dyson series shown in Fig. 7 [34,44].[25] The sum of these diagrams (in an arbitrary R_ξ gauge) can be written as

$$\begin{aligned}
S_{[\mathrm{Fig.7}]} = \frac{i}{2}\int \mathrm{d}^4x\,\mathbf{G}_\varphi\,\bar{\boldsymbol{\psi}}^{a_1}\left\{Y_{za_1a_2}\,P_L + Y^\star_{za_1a_2}\,P_R\right\}\boldsymbol{\psi}^{a_2}\,\times\\
\times\frac{1}{0-\xi\,m_Z^2}\left\{\widetilde{\Gamma}_{mz}(0,0)_{[1.D]} + \xi\,m_Z\,P_{Zm}(0)_{[5.D]}\right\}\,,
\end{aligned} \tag{76}$$

where the m index corresponds to the $\phi^m \equiv G_\varphi$ field, while $P_{Zm}(0)_{[5.D]}$ parametrizes the mixed Z_ν-G_φ two-point function

$$\widetilde{\Gamma}^{\ \nu}_{m\alpha}(q,-q)_{[5.D]} = i\,q^\nu\,P_{\alpha m}(q^2)_{[5.D]}\,,$$

with $A^\alpha_\nu \equiv Z_\nu$. The subscripts $_{[1.D]}$ and $_{[5.D]}$ indicate that in the present model the mixed self-energies are produced by the fermionic loops (Figs. 1.D and 5.D). The explicit expression for

[25] The external line corrections are the sole source of the $\mathcal{O}(\hbar)$ couplings between the Majoron and quarks. By contrast, proper vertex corrections contribute at $\mathcal{O}(\hbar)$ to the couplings between the Majoron and charged leptons [34,44].

$\widetilde{\Gamma}_{m\alpha}^{\ \nu}(q,-q)_{[5.D]}$ is given in Eq. (58); see also the remarks below Eq. (59). To obtain Eq. (76) we have used the fact that $\boldsymbol{\psi}^a(x)$ satisfies the free equations of motions with (up to negligible corrections) the tree-level mass matrices M_F and $M_F^\star$, which are related to the Yukawa matrices by the gauge invariance

$$\mathfrak{f}_\alpha^\top M_F + M_F\,\mathfrak{f}_\alpha = -(\mathcal{T}_\alpha v_{(0)})^j\, Y_j\,,$$

were $(\mathcal{T}_\alpha v_{(0)})^j = m_Z\,\delta^j_{\ z}$ for $\alpha = Z$.

We have $\widetilde{\Gamma}_{mz}(0,0)_{[1.D]} = 0$, while the "Ward identity" (59) gives (cf. Eq. (73))

$$P_{Zm}(0)_{[5.D]} = -\hbar\, m_Z\,\mathbf{b}\,,$$

(the Majoron $G_\varphi = \phi^m$ is gauge-sterile and therefore $[\mathcal{T}_\alpha]^j_{\ m} \equiv 0$). Thus, Eq. (76) agrees with the result of the properly generalized LSZ prescription in which the S-matrix elements are always extracted from (completely) amputated correlation functions.

5. Derivation of the prescription for vector fields

In this section we carefully investigate the structure of propagators of a system of mixed vector (gauge) and scalar fields using the relevant Slavnov–Taylor identities, thereby justifying the practical prescriptions given in Section 2.2. For completeness we construct the corresponding asymptotic fields (which enter the formula (13) for the S-operator) including also those terms which create/annihilate particle states which are not physical (in the sense of the BRST classification).

5.1. Slavnov–Taylor identities

We begin by recalling the identities satisfied by the renormalized 1PI effective action Γ of a non-anomalous gauge theory. Firstly, it must obey the Zinn-Justin identity [45,40,41] (see also [39])

$$\mathcal{S}(\Gamma) = 0\,, \tag{77}$$

in which $\mathcal{S}(F)$ for an arbitrary functional F of fields and antifields is given by (cf. Eq. (39))

$$\mathcal{S}(F) \equiv \frac{\delta F}{\delta K^\mu_\alpha}\cdot\frac{\delta F}{\delta A^\alpha_\mu} + \frac{\delta F}{\delta K_i}\cdot\frac{\delta F}{\delta\phi^i} + \frac{\delta F}{\delta\bar{K}_a}\cdot\frac{\delta F}{\delta\psi^a} + \frac{\delta F}{\delta L_\alpha}\cdot\frac{\delta F}{\delta\omega^\alpha} + h_\alpha\cdot\frac{\delta F}{\delta\overline{\omega}_\alpha}\,.$$

(We use here the abbreviated notation $k\cdot g \equiv \int d^4x\; k(x)\,g(x)$.) In the lowest order, $\Gamma = I_0 + \mathcal{O}(\hbar)$ and $\mathcal{S}(I_0) = 0$ is nothing but the condition of BRST-invariance of the tree-level action (38). Secondly, Γ satisfies also the auxiliary identities: the gauge condition identity and the ghost identity [39] which in the Landau gauge take the forms[26]

$$\frac{\delta\Gamma}{\delta h_\beta(x)} = -\partial^\nu A^\beta_\nu(x)\,, \tag{78}$$

$$\left\{\frac{\delta}{\delta\overline{\omega}_\alpha(x)} - \frac{\partial}{\partial x_\mu}\frac{\delta}{\delta K^\mu_\alpha(x)}\right\}\Gamma = 0\,. \tag{79}$$

[26] These two identities generalize (in different forms) also to other gauge conditions.

Finally, Γ satisfies also the antighost identity

$$\int \mathrm{d}^4x \left\{ \frac{\delta}{\delta\omega^\alpha(x)} - \overline{\omega}_\gamma(x) e^\gamma{}_{\alpha\beta} \frac{\delta}{\delta h_\beta(x)} \right\} \Gamma = \int \mathrm{d}^4x \left\{ L_\beta e^\beta{}_{\alpha\gamma}\omega^\gamma - K^\mu_\beta e^\beta{}_{\alpha\gamma} A^\gamma_\mu + \right.$$

$$\left. - K_i\, [\mathcal{T}_\alpha(\phi + v)]^i + \bar{K}_a([\mathfrak{f}_\alpha]^a{}_b P_L + [\mathfrak{f}^\star_\alpha]^a{}_b P_R)\psi^b \right\}, \tag{80}$$

which is specific only for the Landau gauge [22].[27]

The identities (77)–(80) differentiated with respect to fields give, after restriction to vanishing configurations of fields (cf. Eq. (4)), relations between two-point functions which will be useful in investigation of propagators (in general, relations originating from the Zinn-Justin identity are usually called the Slavnov–Taylor identities (STids) [40,41]). Before we write down the relevant identities, we need to parameterize the 1PI two-point functions. Those of scalar and vector fields are parametrized as in (5) and (17) (cf. Eq. (3)). The mixed, vector-scalar correlation function $\left\langle \hat{A}^\alpha_\mu(-q)\hat{\phi}^j(q) \right\rangle$ is written as

$$\widetilde{\Gamma}^\mu_{\alpha j}(-q,q) \equiv i\, q^\mu\, P_{\alpha j}(q^2) = -\widetilde{\Gamma}^{\ \mu}_{j\alpha}(-q,q)\,. \tag{81}$$

The correlation functions of Nakanishi–Lautrup multipliers: $\left\langle \hat{h}_\alpha(-q)\hat{A}^\beta_\nu(q) \right\rangle$, $\left\langle \hat{h}_\alpha(-q)\hat{\phi}^j(q) \right\rangle$ and $\left\langle \hat{h}_\alpha(-q)\hat{h}_\beta(q) \right\rangle$, are uniquely fixed to all orders by the gauge condition (78)

$$\begin{aligned} &\widetilde{\Gamma}^{\alpha\nu}_{\ \ \beta}(-q,q) \equiv i\,\delta^\alpha_{\ \beta}\, q^\nu = -\widetilde{\Gamma}^{\nu\alpha}_{\ \ \beta}(-q,q)\,, \\ &\widetilde{\Gamma}^\alpha{}_j(-q,q) \equiv 0 = \widetilde{\Gamma}_j{}^\alpha(-q,q)\,, \\ &\widetilde{\Gamma}^{\alpha\beta}(-q,q) \equiv 0\,. \end{aligned} \tag{82}$$

The correlation functions of antifields are parametrized as

$$\frac{\delta}{\delta\hat{\omega}^\gamma(p)} \frac{\delta}{\delta\hat{K}_i(q)} \Gamma \bigg|_0 = (2\pi)^4 \delta^{(4)}(q+p)\, B(q^2)^i{}_\gamma\ , \tag{83}$$

$$\frac{\delta}{\delta\hat{\omega}^\gamma(p)} \frac{\delta}{\delta\hat{K}^\mu_\alpha(q)} \Gamma \bigg|_0 = (2\pi)^4 \delta^{(4)}(q+p) \left\{ i\, q_\mu\, \Omega(q^2)^\alpha{}_\gamma \right\}\,. \tag{84}$$

Notice, that the ghost identity (79) ensures that the same matrix $\Omega(q^2)$ appears in the above function and in (28).

We are now ready to write the required STids for the two-point functions (17) and (81). These are

$$P_{\beta j}(q^2)\, B(q^2)^j{}_\gamma = \left\{ q^2 \mathscr{L}_{\alpha\beta}(q^2) + [M^2_V(q^2) - q^2\mathbb{1}]_{\alpha\beta} \right\} \Omega(q^2)^\alpha{}_\gamma\,, \tag{85}$$

and

[27] In theories with Abelian ideals, additional auxiliary identities are satisfied [46,47]. They encode the lack of certain quantum corrections (e.g. they enforce the vanishing of a determinant of the matrix in Eq. (71)), and therefore play an important role in the proof of renormalizability of non-semisimple gauge-models [46,47]. Nonetheless, we do not use them in what follows: we treat Abelian gauge fields on an equal footing with non-Abelian ones.

$$q^2\, P_{\alpha j}(q^2)\, \Omega(q^2)^{\alpha}{}_{\gamma} = [q^2\mathbb{1} - M_S^2(q^2)]_{ij}\, B(q^2)^i{}_{\gamma}\,. \tag{86}$$

Moreover, from the antighost identity (80) it follows that

$$B(0)^i{}_{\gamma} = (\mathcal{T}_{\gamma} v)^i\,, \tag{87}$$

where v is the (exact) vacuum expectation value (16).

These relations allow us to prove the prescription (31) for massless eigenvectors $\zeta_{V[\lambda_r]}$: since the 1PI functions in four dimensions do not have poles,[28] we see that Eq. (31) follows immediately from (85) after contracting both sides with a vector $\Theta = (\Theta^{\gamma})$ obeying Eq. (30). Similarly, combining (86) with (87), one immediately obtains the Goldstone theorem:

$$M_S^2(0)_{ij}(\mathcal{T}_{\gamma} v)^j = 0\,. \tag{88}$$

It is also worth noticing, that in the Landau gauge there are no one-loop diagrams contributing to the two-point function (83); therefore

$$B(q^2)^i{}_{\gamma} = (\mathcal{T}_{\gamma} v_{(0)})^i + \hbar(\mathcal{T}_{\gamma} v_{(1)})^i + \mathcal{O}(\hbar^2)\,, \tag{89}$$

in agreement with (87). Thus, the STids (85)–(86), together with the invertibility of $\Omega(q^2)$ (cf. Eq. (55)) allow us to express the form-factors $\mathscr{L}_{\alpha\beta}(q^2)$ and $P_{\alpha j}(q^2)$ at one-loop order in terms of quantities which at one-loop have been explicitly calculated in Sec. 3.

5.2. *Propagators*

Inverting the complete matrix of the 1PI two-point functions $\widetilde{\Gamma}$ whose different blocks have been parametrized in the previous section, that is solving the algebraic equation

$$\widetilde{\Gamma}_{IJ}(-p,p)\,\widetilde{G}^{JK}(p,-p) = i\,\delta_I^{\,K}\,, \tag{90}$$

we find the matrix $\widetilde{G}$ of propagators with (resumed) quantum corrections The indices I, J and K run here over components of bosonic fields ϕ^n, A^{α}_{μ} and h_{β}. The resulting expressions for the $\phi\phi$ and AA propagators are given by the formulae (6) and (19), respectively. The mixed scalar-vector propagators vanish, as has been already said (see (18)). The propagators which mix the Nakanishi–Lautrup fields h_{β} with vectors, scalars and themselves have the form

$$\begin{aligned}
&\widetilde{G}_{\beta\mu}^{\ \alpha}(q,-q) = -\delta^{\alpha}{}_{\beta}\,\frac{q_{\mu}}{q^2} = -\widetilde{G}^{\ \alpha}_{\mu\beta}(q,-q)\,, \\
&\widetilde{G}_{\beta}{}^{n}(q,-q) = i\,P_{\beta j}(q^2)\left[\left(q^2\mathbb{1} - M_S^2(q^2)\right)^{-1}\right]^{jn} = \widetilde{G}^{n}{}_{\beta}(q,-q)\,, \\
&\widetilde{G}_{\beta\gamma}(q,-q) = i\left\{P_{\beta n}(q^2)\left[\left(q^2\mathbb{1} - M_S^2(q^2)\right)^{-1}\right]^{nj} P_{\gamma j}(q^2) + \right. \\
&\qquad\qquad \left. +\delta_{\beta\gamma} - \frac{1}{q^2}M_V^2(q^2)_{\beta\gamma} - \mathscr{L}_{\beta\gamma}(q^2)\right\}.
\end{aligned} \tag{91}$$

The last two propagators can be simplified by exploiting the STids (85)–(86) which lead to

[28] This statement is correct in finite orders of perturbation theory.

$$\widetilde{G}_{\beta}{}^{n}(q,-q) = \frac{i}{q^2}\, B(q^2)^{n}{}_{\gamma} \left[\Omega(q^2)^{-1}\right]^{\gamma}_{\ \beta} = \widetilde{G}^{\,n}{}_{\beta}(q,-q)\,, \tag{92}$$

$$\widetilde{G}_{\beta\gamma}(q,-q) = 0\,. \tag{93}$$

Finally, the ghost–antighost propagator has the form

$$\widetilde{\mathcal{G}}^{\beta}_{\ \alpha}(q,-q) = -\frac{i}{q^2}\left[\Omega(q^2)^{-1}\right]^{\beta}_{\ \alpha}\,, \tag{94}$$

where the matrix $\Omega(q^2)$ is defined by (28) or, equivalently, by (84).

5.3. Pole structure of the propagators

The first step in finding the asymptotic fields that appear in the LSZ formula (13) for the S-operator is to determine the behavior of all propagators of the theory in the vicinity of their singularities located on the real axis [6]. As we have already said in Sec. 2.2, the discussion of infrared divergences is beyond the scope of the present paper. Therefore we assume that an IR regulator has been introduced, if necessary, so that the limits (23) and (26) (as well as the $\Omega(0)$ matrix in Eq. (28)) are finite.

With this proviso,[29] from (94) we immediately obtain the pole part of the (anti)ghosts propagator

$$\widetilde{\mathcal{G}}^{\beta}_{\ \alpha}(q,-q)_{\rm pole} = -\frac{i}{q^2}\left[\Omega(0)^{-1}\right]^{\beta}_{\ \alpha}\,. \tag{95}$$

Similarly, the formulae (91)–(93) give the near-pole behavior of the nontrivial propagators of the Nakanishi–Lautrup fields:

$$\widetilde{G}^{\ \alpha}_{\beta\mu}(q,-q)_{\rm pole} = -\widetilde{G}^{\alpha}_{\ \mu\beta}(q,-q)_{\rm pole} = -\delta^{\alpha}_{\ \beta}\,\frac{q_\mu}{q^2}\,, \tag{96}$$

$$\begin{aligned}\widetilde{G}_{\beta}{}^{n}(q,-q)_{\rm pole} = \widetilde{G}^{\,n}{}_{\beta}(q,-q)_{\rm pole} &= \frac{i}{q^2}\, B(0)^{n}{}_{\gamma}\left[\Omega(0)^{-1}\right]^{\gamma}_{\ \beta} = \\ &= \frac{i}{q^2}\,(\mathcal{T}_\gamma\, v)^{n}\left[\Omega(0)^{-1}\right]^{\gamma}_{\ \beta}\,.\end{aligned} \tag{97}$$

Obviously, the pole part of the hh propagator $\widetilde{G}_{\beta\gamma} \equiv 0$, as well as of the mixed scalar-vector propagator $\widetilde{G}^{\ j\beta}_{\ \nu} = \widetilde{G}^{\ \beta j}_{\ \nu} = 0$, vanish. The relevant behavior of the scalar fields propagator can be obtained directly from its form (10):

$$\widetilde{G}^{kj}(p,-p)_{\rm pole} = {\sum_{\ell}}' \sum_{r} \zeta^{k}_{S[\ell_r]}\,\frac{i}{p^2 - m^2_{S(\ell)}}\,\zeta^{j}_{S[\ell_r]}\,, \tag{98}$$

(recall that the prime indicates restriction of the summation to the poles at real values of $p^2 = m^2_{S(\ell)}$). As explained in Sec. 2.1, the corresponding coefficients $\zeta^{k}_{S[\ell_r]}$ can be chosen to be real; we assume that this choice has been made here.

[29] As we have seen at the end of Sec. 4.1, the $\Omega(0)$ matrix is IR-divergent in QCD; therefore Eq. (95) illustrates the need for an IR regulator.

It remains to investigate the propagators (19) of the vector fields. Clearly, all poles of (20) located at real values of q^2 should be taken into account. Just as in (98) we assume that vectors $\zeta^{\beta}_{V[\lambda_r]}$ corresponding to these poles in (20) have been chosen to be real. Moreover, it will be convenient to single out the pole located at $q^2 = 0$ and to label it by $\lambda = \mathbf{0}$. The behavior of the propagator (19) near its real poles can be then written in the form

$$\widetilde{G}^{\beta\delta}_{\nu\rho}(q,-q)_{\text{pole}} = -i\sum_{\lambda\neq\mathbf{0}}{}' \left[\eta_{\nu\rho} - \frac{q_\nu q_\rho}{m^2_{V(\lambda)}}\right]\frac{1}{q^2 - m^2_{V(\lambda)}}\sum_r \zeta^{\beta}_{V[\lambda_r]}\zeta^{\delta}_{V[\lambda_r]} + \\ -\frac{i}{q^2}\,\eta_{\nu\rho}\,\mathcal{Z}^{\beta\delta} + i\,\frac{q_\nu\, q_\rho}{q^2}\,\mathcal{R}^{\beta\delta} + i\,\frac{q_\nu\, q_\rho}{(q^2)^2}\,\mathcal{Z}^{\beta\delta}\,, \tag{99}$$

in which

$$\mathcal{Z}^{\beta\delta} = \sum_r \zeta^{\beta}_{V[\mathbf{0}_r]}\zeta^{\delta}_{V[\mathbf{0}_r]}\,, \tag{100}$$

and $\mathcal{R}$ is given by the formulae (113)–(114) below. The remainder of this section is devoted to the derivation of (99). Construction of the asymptotic states corresponding to the propagators in Eqs. (95)–(99) is given in Sections 5.4 and 5.5.

Let us start with the equality (20). If the limit $q^2 \to m^2_{V(\lambda)}$ of $M^{2\prime}_V(q^2)$ is finite, the form of the right hand side of (20) follows immediately from the analysis of the scalar fields propagator carried out in [19]. However in Sec. 4.1, we have encountered a physically important example in which some of the matrix elements of $M^{2\prime}_V(q^2)$ were IR divergent. Therefore, as proposed in Sec. 2.2, we will only assume that there exists a finite limit (23) for each eigenvector ξ of the matrix $M^2_V(m^2_{V(\lambda)})$ associated with its eigenvalue $m^2_{V(\lambda)}$. This requires a slight modification of the reasoning presented in [19].

We first need some facts proved in [19]. Let

$$R_\lambda(s) = \left(s\mathbb{1} - M^2_V(m^2_{V(\lambda)})\right)^{-1},$$

($s \equiv q^2$) be a resolvent of $M^2_V(m^2_{V(\lambda)})$. Assuming that each generalized eigenvector (see e.g. [36]) of $M^2_V(m^2_{V(\lambda)})$ associated with the eigenvalue $m^2_{V(\lambda)}$ is an (ordinary) eigenvector, and using the explicit form [19] of $R_\lambda(s)$ written in the Jordan basis of $M^2_V(m^2_{V(\lambda)})$ we can write

$$(s - m^2_{V(\lambda)})\,R_\lambda(s) = \mathbb{P}(\lambda) + (s - m^2_{V(\lambda)})F_\lambda(s)\,, \tag{101}$$

where $F_\lambda(s)$ has for $s \to m^2_{V(\lambda)}$ a finite limit $F_\lambda(m^2_{V(\lambda)})$, while $\mathbb{P}(\lambda)$ is the projection onto the eigenspace of $M^2_V(m^2_{V(\lambda)})$ corresponding to its eigenvalue $m^2_{V(\lambda)}$ along the direct sum of remaining generalized eigenspaces of $M^2_V(m^2_{V(\lambda)})$.[30] As was shown in [19], the projection $\mathbb{P}(\lambda)$ can be written as the sum of products

$$\mathbb{P}(\lambda) = \sum_r \xi_{[\lambda_r]}\xi^{\top}_{[\lambda_r]}\,, \tag{102}$$

where the vectors $\{\xi_{[\lambda_r]}\}$ form a basis of the eigenspace corresponding to $m^2_{V(\lambda)}$ and fulfill the following normalization conditions

[30] The decomposition (101) is obvious if $M^2_V(m^2_{V(\lambda)})$ is a diagonalizable matrix.

$$\xi_{[\lambda_r]}^{\top}\xi_{[\lambda_s]} = \delta_{rs}\,.$$

Now, let $\mathscr{A}_\lambda(s)$ be a matrix such that

$$M_V^2(s) = M_V^2(m_{V(\lambda)}^2) + (s - m_{V(\lambda)}^2)\mathscr{A}_\lambda(s)\,. \tag{103}$$

Applying the Lagrange's mean value theorem to the matrix elements of $M_V^2(s)\mathbb{P}(\lambda)$, we see that $\mathscr{A}_\lambda(s)$ has the following property

$$\lim_{s\to m_{V(\lambda)}^2}\left\{\mathscr{A}_\lambda(s)\mathbb{P}(\lambda)\right\} = \lim_{s\to m_{V(\lambda)}^2}\left\{M_V^{2\,\prime}(s)\mathbb{P}(\lambda)\right\} \equiv \mathscr{G}_\lambda\,, \tag{104}$$

because the second limit is finite by our assumptions.

It is convenient to denote

$$\Big(s\mathbb{1} - M_V^2(s)\Big)^{-1} \equiv R_{\text{tot}}(s)\,. \tag{105}$$

We have the obvious equality

$$R_{\text{tot}}(s) = R_\lambda(s)\Big\{\mathbb{1} - (s - m_{V(\lambda)}^2)\mathscr{A}_\lambda(s)\,R_\lambda(s)\Big\}^{-1}\,, \tag{106}$$

from which it follows that

$$\lim_{s\to m_{V(\lambda)}^2}\Big\{(s - m_{V(\lambda)}^2)R_{\text{tot}}(s)\Big\} = \mathbb{P}(\lambda)\big\{1 - \mathscr{G}_\lambda\big\}^{-1}\,. \tag{107}$$

It is also easy to check that

$$\mathbb{P}(\lambda)\big\{\mathbb{1} - \mathscr{G}_\lambda\big\}^{-1} = \sum_r \zeta_{V[\lambda_r]}\zeta_{V[\lambda_r]}^{\top}\,, \tag{108}$$

where the vectors

$$\zeta_{V[\lambda_r]} = \sum_s \mathcal{N}(\lambda)^s{}_r\,\xi_{[\lambda_s]}\,, \tag{109}$$

form a basis of the eigenspace and obey the normalization condition (25). This completes the derivation of the general decomposition (20).

The decomposition (20) is all we need to obtain the behavior of propagator (19) near its poles at $q^2 \neq 0$. Poles located at $q^2 = 0$ require, however, a refined treatment because of the factor $q_\mu q_\nu/q^2$. Namely, we have to show that

$$s\,R_{\text{tot}}(s) = \sum_r \zeta_{V[\mathbf{0}_r]}\zeta_{V[\mathbf{0}_r]}^{\top} + s\,\mathcal{R} + s\,\mathscr{R}(s)\,, \tag{110}$$

where $\mathscr{R}(s) \to 0$ for $s \to 0$. If (110) holds, it will directly lead to the decomposition (99). To ensure that Eq. (110) does indeed hold, we need to assume that the limit

$$\mathcal{B}_0 = \frac{1}{2}\lim_{s\to 0}\Big\{M_V^{2\,\prime\prime}(s)\;\mathbb{P}(\mathbf{0})\Big\}\,, \tag{111}$$

is finite. The Taylor's theorem then implies that

$$M_V^2(s)\,\mathbb{P}(\mathbf{0}) = M_V^2(0)\,\mathbb{P}(\mathbf{0}) + s\,\mathscr{G}_{\mathbf{0}} + s^2\,\mathcal{B}(s)\,, \tag{112}$$

where $\mathscr{G}_\mathbf{0}$ is the limit defined (for $\lambda = \mathbf{0}$) by (104), while $\mathcal{B}(s) \to \mathcal{B}_0$ when $s \to 0$. Moreover, for $s = 0$ the imaginary parts of all Feynman diagrams contributing to the two-point 1PI function vanish which implies that the symmetric matrix $M_V^2(0)$ is always real and, consequently, diagonalizable. In particular, the equality (101) takes then the form

$$R_\mathbf{0}(s) = \frac{1}{s}\,\mathbb{P}(\mathbf{0}) + F_\mathbf{0}(s)\,,$$

with

$$F_\mathbf{0}(s) = \sum_{M\neq 0} (s - M)^{-1} P_M\,,$$

where M runs over (different) nonzero eigenvalues of $M_V^2(0)$ and P_M is the projection onto the eigenspace associated with M along the direct sum of remaining eigenspaces of $M_V^2(0)$. Defining now

$$\mathcal{X}(s) = -(\mathbb{1} - \mathscr{G}_\mathbf{0}^\top)^{-1}\big\{s\,\mathcal{B}(s)^\top + F_\mathbf{0}(s)\big[M_V^2(s) - M_V^2(0)\big]\big\}\,,$$

(clearly, $\mathcal{X}(s) \to 0$ for $s \to 0$), and using the relation

$$\mathbb{P}(\mathbf{0})(\mathbb{1} - \mathscr{G}_\mathbf{0})^{-1} = (\mathbb{1} - \mathscr{G}_\mathbf{0}^\top)^{-1}\,\mathbb{P}(\mathbf{0})\,,$$

(which is an immediate consequence of the relation (108)), one can prove the following identity

$$\begin{aligned} s\,R_{\rm tot}(s) = {} & \mathbb{P}(\mathbf{0})(\mathbb{1} - \mathscr{G}_\mathbf{0})^{-1} + s\,(\mathbb{1} - \mathscr{G}_\mathbf{0}^\top)^{-1}\,F_\mathbf{0}(s) + \\ & + s\,(\mathbb{1} + \mathcal{X}(s))^{-1}(\mathbb{1} - \mathscr{G}_\mathbf{0}^\top)^{-1}\big\{\mathcal{B}(s)^\top\mathbb{P}(\mathbf{0}) + F_\mathbf{0}(s)\mathscr{G}_\mathbf{0}\big\}(\mathbb{1} - \mathscr{G}_\mathbf{0})^{-1} + \\ & + s^2\,(\mathbb{1} + \mathcal{X}(s))^{-1}(\mathbb{1} - \mathscr{G}_\mathbf{0}^\top)^{-1}F_\mathbf{0}(s)\mathcal{B}(s)(\mathbb{1} - \mathscr{G}_\mathbf{0})^{-1} + \\ & - s\,(\mathbb{1} + \mathcal{X}(s))^{-1}\mathcal{X}(s)(\mathbb{1} - \mathscr{G}_\mathbf{0}^\top)^{-1}F_\mathbf{0}(s)\,. \end{aligned}$$

The last two terms tend to zero faster than s, what gives us the decomposition (110); looking at the $\mathcal{O}(s)$ terms we obtain the following formula for $\mathcal{R}$

$$\mathcal{R} = (\mathbb{1} - \mathscr{G}_\mathbf{0}^\top)^{-1}\,F_\mathbf{0}(0)\,(\mathbb{1} - \mathscr{G}_\mathbf{0})^{-1} + (\mathbb{1} - \mathscr{G}_\mathbf{0}^\top)^{-1}\,\mathcal{B}_0^\top\mathbb{P}(\mathbf{0})\,(\mathbb{1} - \mathscr{G}_\mathbf{0})^{-1}\,.$$

Notice that the matrix $\mathcal{R}$ is symmetric (cf. Eq. (111)), as it should be. For future reference, we rewrite this formula in a simpler form. To this end, we note that (104) and (108), together with the definition (100) of the $\mathcal{Z}$ matrix, give

$$\mathcal{R} = -\big[(\mathbb{1} - \mathscr{G}_\mathbf{0})^{-1}\big]^\top\Big\{\sum_{M\neq 0} M^{-1}P_M\Big\}(\mathbb{1} - \mathscr{G}_\mathbf{0})^{-1} + \frac{1}{2}\lim_{s\to 0}\Big[\mathcal{Z}M_V^{2\,\prime\prime}(s)\mathcal{Z}\Big]\,, \tag{113}$$

where $(\mathbb{1} - \mathscr{G}_\mathbf{0})^{-1}$ can be represented as

$$(\mathbb{1} - \mathscr{G}_\mathbf{0})^{-1} = \mathbb{1} + \lim_{s\to 0}[M_V^{2\,\prime}(s)\,\mathcal{Z}]\,. \tag{114}$$

5.4. Propagators with non-simple poles

As follows from the formulae (99)–(100), in the Landau gauge the propagators of vector fields have second order poles if in the particle spectrum of the considered theory massless spin 1 particles are present. Therefore, as the first step, we explicitly construct in this section the generic free field operator whose time-ordered propagator has second order poles. In the second step, the asymptotic states of a general renormalizable model are reconstructed in Sec. 5.5 on the basis of the structure of real poles of the theory propagators (95)–(99).

Consider a set of annihilation and creation operators satisfying the following (anti)commutation relations

$$\left[a_A(\mathbf{p}),\ a_B(\mathbf{p}')^\dagger\right]_\mp = g_{AB}\, 2E_A(\mathbf{p})\,(2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{p}')\,,$$
$$\left[a_A(\mathbf{p}'),\ a_B(\mathbf{p})\right]_\mp = 0\,, \tag{115}$$

with upper and lower signs for bosons and fermions, respectively. Here the labels A, B, etc. distinguish different states $a^\dagger_A(\mathbf{p})|0\rangle$ with the same momentum $\mathbf{p}$; $E_A(\mathbf{p}) = \sqrt{m_A^2 + \mathbf{p}^2}$ is the energy of the state, while $g_{AB} = g^\star_{BA}$ is a matrix that determines the scalar product in the pseudo-Fock space (see e.g. [48]). We assume that

$$g_{AB} = 0\,, \qquad \text{if} \qquad m_A \neq m_B\,,$$

and that $g_{AB} \neq 0$ only if both states A and B are bosonic or both are fermionic.

Out of the operators $a_A(\mathbf{p})$, $a^\dagger_A(\mathbf{p})$ one can construct free fields

$$\Psi^I(x) \equiv \Psi^I_{(-)}(x) + \Psi^I_{(+)}(x)\,, \tag{116}$$

where

$$\Psi^I_{(-)}(x) = \sum_A \int \frac{d^3\mathbf{k}}{(2\pi)^3}\left\{\frac{U^I_A(\mathbf{k})}{2E_A(\mathbf{k})} + i\,x_0\,\frac{R^I_A(\mathbf{k})}{4E_A(\mathbf{k})^2}\right\}\exp(-i\,\bar{k}x)a_A(\mathbf{k})\,, \tag{117}$$

and

$$\Psi^I_{(+)}(x) = \sum_A \int \frac{d^3\mathbf{k}}{(2\pi)^3}\left\{\frac{V^I_A(\mathbf{k})}{2E_A(\mathbf{k})} - i\,x_0\,\frac{S^I_A(\mathbf{k})}{4E_A(\mathbf{k})^2}\right\}\exp(+i\,\bar{k}x)a_A(\mathbf{k})^\dagger\,. \tag{118}$$

$\bar{k}$ denotes here the on-shell four-momentum, $\bar{k} = (\bar{k}^\mu) = (E_A(\mathbf{k}),\ \mathbf{k})$, and U, R, S and V are certain functions. The non-exponential dependence on time x_0 implies [6] that the Fourier transform $\hat{\Psi}^I(q)$ contains, in addition to the delta function $\delta(q^2 - m_A^2)$, also its derivative $\delta'(q^2 - m_A^2)$. Such a time-dependence is characteristic of a non-diagonalizable (*pseudo*Hermitian) Hamiltonian [48].

The time-ordered propagator

$$\mathbf{G}^{IJ}(x,y) = \langle T(\Psi^I(x)\Psi^J(y))\rangle = \Theta(x^0 - y^0)\,\langle 0|\Psi^I(x)\Psi^J(y)|0\rangle +$$
$$\pm\Theta(y^0 - x^0)\,\langle 0|\Psi^J(y)\Psi^I(x)|0\rangle, \tag{119}$$

(with the upper and lower signs corresponding to bosons and fermions, respectively) can be easily found by applying the standard textbook procedure [3]. In particular, the Θ functions can be traded for an integral over an independent time component k^0 of the momentum. In order that the explicit time factors do not spoil the translational invariance of the propagator, the functions U, R, S and V have to satisfy, (for each value of mass m) the following consistency conditions

$$\sum_A^{(m)}\sum_B^{(m)} \Big\{ R^I_A(\mathbf{k})\, g_{AB}\, V^J_B(\mathbf{k}) - U^I_A(\mathbf{k})\, g_{AB}\, S^J_B(\mathbf{k}) \Big\} = 0\,, \tag{120}$$

$$\sum_A^{(m)}\sum_B^{(m)} R^I_A(\mathbf{k})\, g_{AB}\, S^J_B(\mathbf{k}) = 0\,, \tag{121}$$

in which the sums run over the indices A and B labeling the states of mass m. If these conditions are satisfied, the propagator $\mathbf{G}^{IJ}(x,y)$ still contains explicit factors of time, but only in the combination $(x^0 - y^0)$ which can be eliminated by integrating by parts. It is this operation which gives rise to second order poles in the momentum space propagator $\widetilde{\mathbf{G}}^{IJ}(k,-k)$ in

$$\mathbf{G}^{IJ}(x,y) = \int \frac{\mathrm{d}^4 k}{(2\pi)^4}\, e^{-i\,k(x-y)}\, \widetilde{\mathbf{G}}^{IJ}(k,-k)\,, \tag{122}$$

which takes then the form

$$\widetilde{\mathbf{G}}^{IJ}(k,-k) = i \sum_m \left\{ \frac{k^0\, \mathscr{A}^{IJ}_m(\mathbf{k}) + \mathscr{B}^{IJ}_m(\mathbf{k}) - \mathscr{C}^{IJ}_m(\mathbf{k})}{k^2 - m^2 + i\,\varepsilon} + \right.$$
$$\left. -\,2\, \frac{k^0\, \mathscr{D}^{IJ}_m(\mathbf{k}) + E_m(\mathbf{k})^2\, \mathscr{C}^{IJ}_m(\mathbf{k})}{\big[k^2 - m^2 + i\,\varepsilon\big]^2} \right\}, \tag{123}$$

with $E_m(\mathbf{k}) = \sqrt{m^2 + \mathbf{k}^2}$, and

$$\mathscr{A}^{IJ}_m(\mathbf{k}) = \frac{\mathscr{Q}^{(+)IJ}_m(\mathbf{k}) - \mathscr{Q}^{(-)IJ}_m(\mathbf{k})}{2\,E_m(\mathbf{k})}\,,$$

$$\mathscr{B}^{IJ}_m(\mathbf{k}) = \frac{\mathscr{Q}^{(+)IJ}_m(\mathbf{k}) + \mathscr{Q}^{(-)IJ}_m(\mathbf{k})}{2}\,,$$

$$\mathscr{C}^{IJ}_m(\mathbf{k}) = \frac{\mathscr{T}^{(+)IJ}_m(\mathbf{k}) + \mathscr{T}^{(-)IJ}_m(\mathbf{k})}{4\,E_m(\mathbf{k})^2}\,,$$

$$\mathscr{D}^{IJ}_m(\mathbf{k}) = \frac{\mathscr{T}^{(+)IJ}_m(\mathbf{k}) - \mathscr{T}^{(-)IJ}_m(\mathbf{k})}{4\,E_m(\mathbf{k})}\,, \tag{124}$$

where

$$\mathscr{Q}^{(+)IJ}_m(\mathbf{k}) = \sum_A^{(m)}\sum_B^{(m)} U^I_A(\mathbf{k})\, g_{AB}\, V^J_B(\mathbf{k})\,,$$

$$\mathscr{Q}^{(-)IJ}_m(\mathbf{k}) = \pm\sum_A^{(m)}\sum_B^{(m)} U^J_A(-\mathbf{k})\, g_{AB}\, V^I_B(-\mathbf{k}) = \pm\mathscr{Q}^{(+)JI}_m(-\mathbf{k})\,,$$

$$\mathscr{T}^{(+)IJ}_m(\mathbf{k}) = \sum_A^{(m)}\sum_B^{(m)} R^I_A(\mathbf{k})\, g_{AB}\, V^J_B(\mathbf{k})\,,$$

$$\mathscr{T}^{(-)IJ}_m(\mathbf{k}) = \pm\sum_A^{(m)}\sum_B^{(m)} R^J_A(-\mathbf{k})\, g_{AB}\, V^I_B(-\mathbf{k}) = \pm\mathscr{T}^{(+)JI}_m(-\mathbf{k})\,, \tag{125}$$

with upper and lower signs for bosons and fermions, respectively.

5.5. Asymptotic states

We are now in a position to reconstruct the asymptotic states on the basis of the structure (95)–(99) of real poles of the theory propagators, which is the essence of the LSZ asymptotic formalism. Similar analysis was carried out in Ref. [20], where it was applied to some specific gauge theory models. Here we generalize it (in the Landau gauge) to the general case allowing for an arbitrary mixing of fields. The first step is to choose the basis of the subspace of unphysical states. It will be convenient to work with the state vectors $b_\beta(\mathbf{k})^\dagger|0\rangle$ representing the Nakanishi–Lautrup (NL) modes, and the states $d^\beta(\mathbf{k})^\dagger|0\rangle$ of "scalar gauge bosons". Writing now the asymptotic field

$$\mathbf{h}_\beta(x) = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3\, 2|\mathbf{k}|} \left\{ \exp(-i\,\bar{k}x)\, b_\beta(\mathbf{k}) + \exp(+i\,\bar{k}x)\, b_\beta(\mathbf{k})^\dagger \right\}, \tag{126}$$

associated with the Nakanishi–Lautrup multiplier $h_\beta(x)$, in which $\bar{k} = (\bar{k}^\mu) \equiv (|\mathbf{k}|,\ \mathbf{k})$, and taking into account that the propagator of the (interpolating) NL fields has no real poles (in fact it vanishes identically, cf. (93)) we conclude that

$$\left[b_\alpha(\mathbf{p}'),\ b_\beta(\mathbf{p})^\dagger \right]_- = \left[b_\alpha(\mathbf{p}'),\ b_\beta(\mathbf{p}) \right]_- = 0\,. \tag{127}$$

From this it follows that $b_\beta(\mathbf{k})^\dagger|0\rangle$ is a zero-norm state.

Next, we assume the following decomposition of the asymptotic vector field

$$\mathbf{A}^\alpha_\mu = \mathbb{V}^\alpha_\mu + \partial_\mu \mathbb{S}^\alpha + \mathbb{L}^\alpha_\mu\,, \tag{128}$$

in which the scalar field $\mathbb{S}^\alpha$ is built out of the annihilation and creation operators of the "scalar gauge bosons"

$$\mathbb{S}^\alpha(x) = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3\, 2|\mathbf{k}|} \left\{ \exp(-i\,\bar{k}x)\, d^\alpha(\mathbf{k}) + \exp(+i\,\bar{k}x)\, d^\alpha(\mathbf{k})^\dagger \right\}, \tag{129}$$

and the "longitudinal" massless vector field $\mathbb{L}^\alpha_\mu$ involves the operators of the NL zero norm states

$$\mathbb{L}^\alpha_\mu(x) = \mathcal{Z}^{\alpha\beta} \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3\, 2|\mathbf{k}|} \left\{ e^{-i\,\bar{k}x} \left[i\,\frac{(\mathcal{P}\bar{k})_\mu}{4|\mathbf{k}|^2} - \frac{\bar{k}_\mu}{2|\mathbf{k}|} x_0 \right] b_\beta(\mathbf{k}) + \text{H.c.} \right\}, \tag{130}$$

(here $\mathcal{P}\bar{k} = ((\mathcal{P}\bar{k})^\mu) \equiv (|\mathbf{k}|,\ -\mathbf{k})$ denotes the parity transformed momentum $\bar{k}$). The field $\mathbb{V}^\alpha_\mu$ creates and annihilates only the physical states. (Clearly, the creation and annihilation operators of physical states commute with the ones associated with the unphysical states.)

As can be easily checked, the asymptotic fields $\mathbf{h}_\beta$ and $\mathbf{A}^\alpha_\mu$ correctly reproduce the behavior of the mixed propagator (96) in the vicinity of its pole if

$$\left[b_\beta(\mathbf{k}),\ d^\alpha(\mathbf{q})^\dagger \right]_- = \left[d^\alpha(\mathbf{q}),\ b_\beta(\mathbf{k})^\dagger \right]_- = \delta^\alpha_{\ \beta}\, 2|\mathbf{k}|\, (2\pi)^3 \delta^{(3)}(\mathbf{k}-\mathbf{q})\,. \tag{131}$$

Moreover, the pole structure (99) is reproduced by the time-ordered propagator of the asymptotic fields $\mathbf{A}^\alpha_\mu$ provided

$$\left[d^\alpha(\mathbf{k}),\ d^\beta(\mathbf{q})^\dagger \right]_- = \mathcal{R}^{\alpha\beta}\, 2|\mathbf{k}|\, (2\pi)^3 \delta^{(3)}(\mathbf{k}-\mathbf{q})\,, \tag{132}$$

and

$$\mathbb{V}^\alpha_\mu = \sum_\lambda{}' \sum_r \zeta^\alpha_{V[\lambda_r]} \mathbb{A}^{\lambda_r}_\mu\,. \tag{133}$$

(As before, the prime over the first sum indicates that the summation is restricted to indices λ corresponding to poles on the real axis.) Here $\mathbb{A}_\mu^{\lambda_r}$ is the free vector field (in the unitary gauge if $m_{V(\lambda)} \neq 0$, or in the Coulomb gauge, if $m_{V(\lambda)} = 0$) of spin 1 particles of mass $m_{V(\lambda)}$. The $\mathbb{A}_\mu^{\lambda_r}$ field is canonically normalized. For completeness we give here its explicit form[31]

$$\mathbb{A}_\mu^{\lambda_r}(x) = \sum_{\hbar} \int \frac{d^3\mathbf{k}}{(2\pi)^3\, 2\sqrt{m_{V(\lambda)}^2 + \mathbf{k}^2}} \left\{ \exp(-i\,\bar{k}x)\, e_\mu^{\hbar}(\mathbf{k}, m_{V(\lambda)})\, a_{\hbar}^{\lambda_r}(\mathbf{k}) + \text{H.c.} \right\}, \tag{134}$$

with $\bar{k} = (\bar{k}^\mu) \equiv (\sqrt{m_{V(\lambda)}^2 + \mathbf{k}^2},\ \mathbf{k})$, and

$$\left[a_{\hbar}^{\lambda_r}(\mathbf{k}),\ a_{\hbar'}^{\lambda'_{r'}}(\mathbf{q})^\dagger \right]_- = \delta_{\hbar\hbar'}\, \delta_{\lambda\lambda'}\, \delta_{rr'}\, 2\sqrt{m_{V(\lambda)}^2 + \mathbf{k}^2}\, (2\pi)^3 \delta^{(3)}(\mathbf{k} - \mathbf{q})\,,$$

where $\hbar$ and $\hbar'$ run over the helicity values $\pm 1, 0$ (if $m_{V(\lambda)} \neq 0$) or ± 1 (if $m_{V(\lambda)} = 0$). The explicit form of the polarization vectors is

$$e_\mu^-(\mathbf{p}, m) = -e_\mu^+(\mathbf{p}, m)^\star = -\frac{1}{\sqrt{2}\,|\mathbf{p}|\,\sqrt{(p^1)^2 + (p^2)^2}} \begin{bmatrix} 0 \\ p^1 p^3 + i\,p^2|\mathbf{p}| \\ p^2 p^3 - i\,p^1|\mathbf{p}| \\ -(p^1)^2 - (p^2)^2 \end{bmatrix}_\mu ,$$

and

$$e_\mu^0(\mathbf{k}, m) = -\frac{\sqrt{\mathbf{k}^2 + m^2}}{m\,|\mathbf{k}|} \begin{bmatrix} \frac{\mathbf{k}^2}{\sqrt{\mathbf{k}^2 + m^2}} \\ -\mathbf{k} \end{bmatrix}_\mu .$$

It is worth stressing that it is precisely the second line of (99) which fixes the form (130) of the longitudinal field $\mathbb{L}_\mu^\alpha$. In particular, $\mathbb{L}_\mu^\alpha$ is nonzero only if in the theory spectrum there are massless spin 1 particles (cf. the definition (100) of the $\mathcal{Z}$ matrix). For this reason we have called $\mathbb{L}_\mu^\alpha$ the "longitudinal" field: it creates massless gauge bosons of zero helicity (clearly, they form a subspace of the Nakanishi–Lautrup modes).

It remains to construct the asymptotic fields $\boldsymbol{\phi}^i$ associated with the interpolating scalar fields ϕ^i. The decomposition of $\boldsymbol{\phi}^i$ which reproduces the structure (98) of the poles on the real axis of the scalar fields propagator follows immediately from the prescription formulated in Sec. 2.1 and is given by (12). However, it is still necessary to split this asymptotic field into its parts creating/annihilating physical and unphysical states. To this end it is better to forget Eq. (12) altogether and write down the decomposition of $\boldsymbol{\phi}^i$ in terms of fields creating (yet unknown) physical and unphysical states

$$\boldsymbol{\phi}^j = \boldsymbol{\phi}^j_{\text{ph}} + \boldsymbol{\phi}^j_{\text{unph}}\,. \tag{135}$$

Let us introduce the matrix (cf. Eqs. (83)–(84))

$$\widetilde{C}^j{}_\gamma(q^2) = B(q^2)^j{}_\beta\, [\Omega(q^2)^{-1}]^\beta{}_\gamma\,, \tag{136}$$

together with its limit

[31] See e.g. [3]; we use slightly more common normalization conventions, however.

$$C^j_{\ \gamma} = \widetilde{C}^j_{\ \gamma}(0)\,. \tag{137}$$

The structure (97) of the pole of the mixed ϕh propagator and the lack of poles of the (vanishing identically in the Landau gauge) mixed scalar-vector propagator (cf. (18)) are correctly reproduced by $\boldsymbol{\phi}^j_{\text{unph}}$ of (135), if

$$\boldsymbol{\phi}^j_{\text{unph}}(x) = C^j_{\ \gamma}\,\mathbb{S}^\gamma(x) - C^j_{\ \beta}\,\mathcal{R}^{\beta\gamma}\,\mathbf{h}_\gamma(x)\,. \tag{138}$$

Vanishing of the mixed propagator of the asymptotic fields $\boldsymbol{\phi}^j$ and $\mathbf{A}^\alpha_\mu$ (given by (128)), necessary to reproduce (18), hinges on the following relation

$$C^j_{\ \gamma}\,\mathcal{Z}^{\gamma\beta} = 0\,, \tag{139}$$

whose validity can be seen as follows. Let us rewrite the STids (85)–(86) as

$$P_{\beta j}(q^2)\,\widetilde{C}^j_{\ \alpha}(q^2) = \left\{q^2\,\mathscr{L}_{\alpha\beta}(q^2) + [M^2_V(q^2) - q^2\mathbb{1}]_{\beta\alpha}\right\}, \tag{140}$$

and

$$q^2\,P_{\alpha j}(q^2) = [q^2\mathbb{1} - M^2_S(q^2)]_{ij}\,\widetilde{C}^i_{\ \alpha}(q^2)\,. \tag{141}$$

For $q^2 \to 0$ these relations reduce to

$$M^2_V(0)_{\beta\alpha} = P_{\beta j}(0)\,C^j_{\ \alpha}\,, \tag{142}$$

and

$$M^2_S(0)_{ji}\,C^i_{\ \alpha} = 0\,. \tag{143}$$

Now Eq. (141) gives

$$P_{\alpha j}(q^2)\,C^j_{\ \beta} = \frac{1}{q^2}\,\widetilde{C}^i_{\ \alpha}(q^2)\,[q^2\mathbb{1} - M^2_S(q^2)]_{ij}\,C^j_{\ \beta}\,, \tag{144}$$

and using (142)–(143) we get for $q^2 \to 0$

$$M^2_V(0)_{\alpha\beta} = \lim_{q^2\to 0}\left\{C^i_{\ \alpha}\,[\mathbb{1} - M^{2\,\prime}_S(q^2)]_{ij}\,C^j_{\ \beta}\right\}, \tag{145}$$

provided the limit

$$\lim_{q^2\to 0}\left\{M^{2\,\prime}_S(q^2)_{ij}\,C^j_{\ \beta}\right\},$$

exists. Since for $q^2 = 0$ the reality of $M^2_V(q^2)$ cannot be violated, the matrix $M^2_V(0)$ has an orthonormal basis of real eigenvectors $\theta_{(M,n)} = (\theta^\beta_{(M,n)})$, where n distinguishes different eigenvectors $\theta_{(M,n)}$ corresponding to the eigenvalue M. Let us introduce the following set of vectors

$$\zeta^j_{(M,n)} = \frac{1}{\sqrt{M}}\,C^j_{\ \gamma}\,\theta^\gamma_{(M,n)}\,, \qquad \text{for} \quad M \neq 0\,, \tag{146}$$

and

$$\xi^j_{(n)} = C^j_{\ \gamma}\,\theta^\gamma_{(0,n)}\,.$$

Eq. (145) now yields

$$\lim_{q^2\to 0}\left\{\zeta^{\top}_{(M,n)}\left[\mathbb{1}-M^{2\,\prime}_S(q^2)\right]\zeta_{(M',n')}\right\}=\delta_{MM'}\,\delta_{nn'}\,, \tag{147}$$

as well as

$$\lim_{q^2\to 0}\left\{\zeta^{\top}_{(M,n)}\left[\mathbb{1}-M^{2\,\prime}_S(q^2)\right]\xi_{(n')}\right\}=0\,, \tag{148}$$

and

$$\lim_{q^2\to 0}\left\{\xi^{\top}_{(n)}\left[\mathbb{1}-M^{2\,\prime}_S(q^2)\right]\xi_{(n')}\right\}=0\,. \tag{149}$$

Eq. (147) shows that the vectors $\zeta_{(M,n)}$ are linearly independent. Then (148) shows that none of $\xi_{(n')}$ is a linear combination of $\zeta_{(M,n)}$. In fact, Eq. (149) implies that all $\xi_{(n)}$ vanish; this would be obvious if the limit $M^{2\,\prime}_S(0)$ was finite. Indeed, $M^{2\,\prime}_S(q^2)=\mathcal{O}(\hbar)$ and $M^{2\,\prime}_S(q^2)$ is for $q^2<0$ a real symmetric matrix; thus $\mathbb{1}-M^{2\,\prime}_S(0)$ is positive definite (for perturbative values of coupling constants) provided the limit exists. We do not assume finiteness of the whole matrix $M^{2\,\prime}_S(0)$. Nonetheless, finiteness of the limit in Eq. (149) simply means the cancellation of certain $\ln(q^2)$ divergences; therefore Eq. (149) cannot be satisfied for a nonzero vector $\xi_{(n')}=\xi_{(n)}$, at least in the perturbative regime. Hence, we have the following equivalence

$$M^2_V(0)_{\alpha\beta}\Lambda^\beta=0 \qquad\Leftrightarrow\qquad C^i{}_\beta\,\Lambda^\beta=0\,. \tag{150}$$

Recall now that coefficients $\zeta^\beta_{V[\mathbf{0}_r]}$ in the formula (100) for the $\mathcal{Z}^{\beta\delta}$ matrix are null eigenvectors of $M^2_V(0)$; thus we have proved Eq. (139).

Finally we have to consider the scalar-scalar propagators. The time-ordered propagator

$$\langle T(\boldsymbol{\phi}^l_{\text{unph}}(x)\boldsymbol{\phi}^j_{\text{unph}}(y))\rangle=\int\frac{\mathrm{d}^4k}{(2\pi)^4}\,e^{-i\,k(x-y)}\,\widetilde{\mathbf{G}}^{lj}(k,-k)_{\text{unph}}\,, \tag{151}$$

of unphysical fields (138) is easy to find:

$$\widetilde{\mathbf{G}}^{lj}(k,-k)_{\text{unph}}=-\frac{i}{k^2}\,C^l{}_\beta\,C^j{}_\gamma\,\mathcal{R}^{\beta\gamma}\,.$$

The explicit form (113)–(114) of the matrix $\mathcal{R}^{\beta\gamma}$ together with the identity (139) allow us to simplify this expression to

$$\widetilde{\mathbf{G}}^{lj}(k,-k)_{\text{unph}}=\frac{i}{k^2}\,C^l{}_\beta\,C^j{}_\gamma\sum_{M\neq 0}\frac{1}{M}\,(P_M)^{\beta\gamma}\,. \tag{152}$$

Recall that M runs over (different) nonzero eigenvalues of $M^2_V(0)$ while P_M is a projection onto the eigenspace associated with M along the direct sum of the remaining eigenspaces of $M^2_V(0)$; in particular

$$(P_M)^{\beta\gamma}=\sum_n\theta^\beta_{(M,n)}\theta^\gamma_{(M,n)}\,.$$

Comparing (152) rewritten in terms of the vectors (146),

$$\widetilde{\mathbf{G}}^{lj}(k,-k)_{\text{unph}}=\frac{i}{k^2}\sum_{M\neq 0}\sum_n\zeta^l_{(M,n)}\,\zeta^j_{(M,n)}\,,$$

with Eq. (98), one can identify $\zeta_{(M,n)}$ as the eigenvectors $\zeta_{S[\ell_r]}$ associated with the would-be Goldstone bosons. Indeed, the equality (143) shows that $\zeta_{(M,n)}$ are null eigenvectors of $M_S^2(0)$, while (147) implies that they satisfy the (refined version of the) normalization conditions (9), in complete agreement with the general prescription for finding $\zeta_{S[\ell_r]}$ described in Sec. 2.1. The number of vectors (146) equals to the dimension of the gauge group minus the number of massless gauge bosons, as required by the counting of degrees of freedom based on the Goldstone theorem.

Of course, all the eigenvectors $\zeta_{S[\ell_r]}$ corresponding to the same pole at a value $m^2_{S(\ell)}$ of p^2 have to obey the orthogonality conditions (9) in order to ensure the expansion (10) of the propagator.[32] Therefore the physical massless eigenvectors $\zeta_{S[\ell_r]}$ must be orthogonal (with respect to the scalar product (9)) to the unphysical ones (146). This is equivalent to the condition (22) owing to the "non-renormalization theorem" (87). In particular, the physical part $\boldsymbol{\phi}_{\rm ph}$ of the asymptotic scalar field (135) can be written as

$$\boldsymbol{\phi}^j_{\rm ph} = \sum_{\text{phys.}\, r} \zeta^j_{S[\mathbf{0}_r]} \Phi^{\mathbf{0}_r} + \sum_{\ell\neq\mathbf{0}}{}' \sum_r \zeta^j_{S[\ell_r]} \Phi^{\ell_r} \,, \tag{153}$$

where in the first sum corresponding to poles at $p^2 = 0$ (labeled by $\ell = \mathbf{0}$) the summation is over the indices r corresponding to physical eigenvectors $\zeta^j_{S[\mathbf{0}_r]}$, satisfying (for each generator $\mathcal{T}_\alpha$) the following condition

$$\lim_{q^2\to 0} \left\{ \zeta^\top_{S[\mathbf{0}_r]} \left[\mathbb{1} - M_S^{2\prime}(q^2) \right] \mathcal{T}_\alpha v \right\} = 0 \,. \tag{154}$$

As before, the prime over the second sum in (153) indicates that the summation is restricted to the poles located on the real axis. In our conventions the canonically normalized free scalar field Φ^{ℓ_r} of mass $m_{S(\ell)}$ has the form

$$\Phi^{\ell_r}(x) = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3\, 2\sqrt{m^2_{S(\ell)} + \mathbf{k}^2}} \left\{ \exp(-i\,\bar{k}x)\, a^{\ell_r}(\mathbf{k}) + \exp(+i\,\bar{k}x)\, a^{\ell_r}(\mathbf{k})^\dagger \right\} , \tag{155}$$

with $\bar{k} = (\bar{k}^\mu) \equiv (\sqrt{m^2_{S(\ell)} + \mathbf{k}^2},\ \mathbf{k})$, and

$$\left[a^{\ell_r}(\mathbf{k}),\ a^{\ell'_{r'}}(\mathbf{q})^\dagger \right]_- = \delta_{\ell\ell'}\, \delta_{rr'}\, 2\sqrt{m^2_{S(\ell)} + \mathbf{k}^2}\, (2\pi)^3 \delta^{(3)}(\mathbf{k} - \mathbf{q}) \,.$$

The time-ordered propagator of the complete asymptotic scalar field (135)

$$\widetilde{\mathbf{G}}^{lj}(k,-k) = \widetilde{\mathbf{G}}^{lj}(k,-k)_{\rm ph} + \widetilde{\mathbf{G}}^{lj}(k,-k)_{\rm unph} \,,$$

(where $\widetilde{\mathbf{G}}^{lj}(k,-k)_{\rm ph}$ is defined analogously to (151)) matches then the form (98) that the complete scalar fields propagator takes near its poles on the real axis. In particular, the states of physical massless scalars in the pseudo-Fock space are orthogonal to the states of would-be Goldstone modes.

It should be stressed that the asymptotic fields $(\Psi^I) = (\boldsymbol{\phi}^i,\ \mathbf{A}^\alpha_\mu,\ \mathbf{h}_\beta)$ do obey the consistency conditions (120)–(121) and that almost all of their time-ordered propagators $\mathbf{G}^{IJ}(x,y)$, obtained using the general formula (123), indeed exactly reproduce the appropriate expressions

[32] More precisely, as we have shown in Sec. 5.3, only the "refined version" (25) of the normalization conditions is needed for Eq. (20) to hold true. Clearly, the same is true for its scalar counterpart (10).

$G^{IJ}(x, y)_{\text{pole}}$ listed in Eqs. (96)–(99). The sole exception is the propagator of the time components of the vector fields:

$$\widetilde{\mathbf{G}}^{\beta\delta}_{00}(q, -q) = \widetilde{G}^{\beta\delta}_{00}(q, -q)_{\text{pole}} - i\,\mathcal{R}^{\beta\delta} - i\sum_{\lambda\neq\mathbf{0}}{}' \frac{1}{m^2_{V(\lambda)}} \sum_r \zeta^{\beta}_{V[\lambda_r]}\zeta^{\delta}_{V[\lambda_r]}\,.$$

The difference affects only the non-pole parts and therefore is irrelevant for the structure of asymptotic states.

This completes the construction (in the Landau gauge) of the space of asymptotic states in general gauge theories with an arbitrary mixing of fields. It is however worthwhile to show that, also in the presence of generic mixing, the unphysical asymptotic states do have the structure discussed in [6,20] which is required for unitarity of the S-operator restricted to the subspace of physical states. To this end, let us, following [6], introduce the generator of the BRST transformations acting on the asymptotic fields (compare the formulae (40))

$$\begin{aligned}
&i\big[Q_{BRST},\, \boldsymbol{\phi}^j\big]_- = B(0)^j{}_\gamma\,\boldsymbol{\omega}^\gamma\,, && i\big[Q_{BRST},\, \boldsymbol{\psi}^a\big]_+ = 0\,,\\
&i\big[Q_{BRST},\, \mathbf{A}^\alpha_\mu\big]_- = \Omega(0)^\alpha{}_\gamma\,\partial_\mu\boldsymbol{\omega}^\gamma\,, && i\big[Q_{BRST},\, \boldsymbol{\omega}^\alpha\big]_+ = 0\,,\\
&i\big[Q_{BRST},\, \overline{\boldsymbol{\omega}}_\alpha\big]_+ = \mathbf{h}_\alpha, && i\big[Q_{BRST},\, \mathbf{h}_\alpha\big]_- = 0\,,
\end{aligned}\tag{156}$$

with the asymptotic (anti)ghosts fields having the forms

$$\begin{aligned}
\overline{\boldsymbol{\omega}}_\beta(x) &= -i\int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3\,2|\mathbf{k}|}\left\{\exp(-i\,\bar{k}x)\,\overline{b}_\beta(\mathbf{k}) + \exp(+i\,\bar{k}x)\,\overline{b}_\beta(\mathbf{k})^\dagger\right\},\\
\boldsymbol{\omega}^\beta(x) &= \int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3\,2|\mathbf{k}|}\left\{\exp(-i\,\bar{k}x)\,\overline{d}^\beta(\mathbf{k}) + \exp(+i\,\bar{k}x)\,\overline{d}^\beta(\mathbf{k})^\dagger\right\}.
\end{aligned}\tag{157}$$

The only non-vanishing anticommutators of the operators $\overline{b}_\beta(\mathbf{k})$, $\overline{d}_\beta(\mathbf{k})$, etc. are

$$\big[\overline{d}^\alpha(\mathbf{q}),\ \overline{b}_\beta(\mathbf{k})^\dagger\big]_+ = \big[\overline{b}_\beta(\mathbf{k}),\ \overline{d}^\alpha(\mathbf{q})^\dagger\big]_+^\dagger = -i[\Omega(0)^{-1}]^\alpha_\beta\,2|\mathbf{k}|\,(2\pi)^3\delta^{(3)}(\mathbf{k}-\mathbf{q})\,.\tag{158}$$

This ensures that the time-ordered propagator of the ghost fields (157) matches the expression (95).[33] The charge Q_{BRST} is a pseudoHermitian and nilpotent operator. It is easy to check that it can be represented as

$$Q_{BRST} = -i\,\Omega(0)^\alpha{}_\beta\int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3\,2|\mathbf{k}|}\left\{\overline{d}^\beta(\mathbf{k})^\dagger\,b_\alpha(\mathbf{k}) - b_\alpha(\mathbf{k})^\dagger\,\overline{d}^\beta(\mathbf{k})\right\},\tag{159}$$

and thus anticommutation relations (158), as well as Q_{BRST}, have the standard Kugo–Ojima form [20] (up to a redefinition of $\overline{d}^\beta(\mathbf{k})$). We have

$$\begin{aligned}
b_\alpha(\mathbf{k})^\dagger|0\rangle &= Q_{BRST}\,\overline{b}_\alpha(\mathbf{k})^\dagger|0\rangle\,,\\
\overline{d}^\gamma(\mathbf{k})^\dagger|0\rangle &= i\,[\Omega(0)^{-1}]^\gamma_\alpha\,Q_{BRST}\,d^\alpha(\mathbf{k})^\dagger|0\rangle\,,
\end{aligned}\tag{160}$$

and

[33] We note that $\Omega(0)$ is real, because Feynman integrals contributing to $\Omega(q^2)$ cannot acquire imaginary parts for $q^2 = 0$.

$$Q_{BRST}\, b_\alpha(\mathbf{k})^\dagger|0\rangle = 0\,,$$

$$Q_{BRST}\, \overline{d}^\gamma(\mathbf{k})^\dagger|0\rangle = 0\,, \tag{161}$$

which shows that the unphysical states form quartet representations of Q_{BRST}. One has, therefore, the following decomposition [20]

$$\ker Q_{BRST} = \mathscr{F}_{\rm ph} \oplus {\rm im}\, Q_{BRST}\,, \tag{162}$$

in which the subspace $\mathscr{F}_{\rm ph}$ is obtained by the action on the vacuum state $|0\rangle$ of (products of) creation operators $a^{\ell_r}(\mathbf{k})^\dagger$ and $a_{\hbar}^{\lambda_r}(\mathbf{k})^\dagger$ appearing in Eqs. (155) and (134), as well as their spin 1/2 counterparts ("physical particles").[34] The decomposition (162) is obvious in the subspace of one-particle states; by constructing an appropriate family of projection operators [20], one can prove its validity in the entire pseudo-Fock space $\mathscr{F}$. In particular, the scalar product restricted to $\ker Q_{BRST}$ is positive semidefinite (elements of ${\rm im}\, Q_{BRST}$ have a vanishing norm).

Finally, Q_{BRST} commutes [6] with the pseudounitary S-operator

$$S = :\!\exp\left\{-\int {\rm d}^4x\, \Psi^J(x) \int {\rm d}^4y\, \Gamma_{JK}(x,y)\, \frac{\delta}{\delta J_K(y)}\right\}\!: \exp(i\, W[J])\bigg|_{J=0}\,,$$

in which $(\Psi^J) = (\boldsymbol{\phi}^j,\ \mathbf{A}^\alpha_\mu,\ \mathbf{h}_\beta,\ \boldsymbol{\psi}^a,\ \boldsymbol{\omega}^\alpha,\ \overline{\boldsymbol{\omega}}_\alpha)$ now runs over all asymptotic fields (including ghosts).[35] Hence, $\ker Q_{BRST}$ is an invariant subspace for S and the amplitudes between the states belonging to $\mathscr{F}_{\rm ph}$ are consistent with unitarity [6,20].

6. Conclusions

We have shown how the asymptotic approach of Lehmann, Symanzik and Zimmermann to calculating S-matrix elements extends to general gauge theories, treated in the Landau gauge, in the presence of arbitrary mixing of vector (and scalar) fields. The developed formalism covers both exact and spontaneously broken gauge symmetries and takes into account complication arising if there are Goldstone bosons associated with spontaneously broken global symmetries. The pseudo-Fock space of asymptotic states following from the structure of the poles at real values of the momentum variable p^2 (corresponding to stable particles) of the matrix propagators of vector and scalar fields has been explicitly constructed. Its BRST-cohomological structure ensures unitarity of the S-operator restricted to the subspace of physical asymptotic states in the presence of a generic mixing.

On the practical side, a simple prescription, formulated entirely in terms of eigenvectors of certain matrices, for computing "square-rooted residues" ζ of poles of the matrix propagators has been given. It can be viewed as a straightforward generalization of the procedure used to identify fields which are "mass eigenstates" in tree level calculations and can be efficiently used also in numerical or automatized analytical calculations.

These general results, obtained by analyzing the relevant set of Slavnov–Taylor identities, have been supplemented by the ready-to use one-loop formulae for self-energies of vector and

[34] In particular, the states $a^{\mathbf{0}_r}(\mathbf{k})^\dagger|0\rangle$ corresponding to massless eigenvectors $\zeta_{S[\mathbf{0}_r]}$ satisfying the condition (154) belong to $\mathscr{F}_{\rm ph}$. In contrast, states created/annihilated by the $\boldsymbol{\phi}^j_{\rm unph}$ part (explicitly given by (138)) of (135) are unphysical would-be Goldstone modes; they are "confined" in the sense of Kugo–Ojima quartet mechanism [20].

[35] The commutativity with S is a consequence of the Zinn-Justin identity (77); this is why one has to include the $B(0)$ and $\Omega(0)$ factors in the definition (156) of Q_{BRST} [6].

scalar fields valid in any renormalizable gauge theory, and the formulated practical prescriptions have been illustrated on two interesting examples of field mixing.

While the prescription for the ζ factors of the vector fields given in this paper is valid only in the Landau gauge, it can be generalized to other R_ξ gauges, as will be shown in a separate publication.

Finally, although in some reasonings restrictions were made to the perturbative approach (mainly to guarantee the existence of inverses of certain matrices), most of the results should remain valid outside the perturbative expansion as well.

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Appendix A. Results in the singlet Majoron model

In this appendix we list some one-loop results pertaining to the singlet Majoron extension of the SM. Our conventions are described in Sec. 4.2.

The one-loop corrections to the vacuum expectation values ($w_H \equiv v_{H(0)}$ and $w_\varphi \equiv v_{\varphi(0)}$ denote the tree-level VEVs) can be obtained using the general formula (44) and have the form:

$$
\begin{aligned}
v_{H(1)} = \frac{\lambda_2}{(4\pi)^2 w_H^3 \left(\lambda_1\lambda_2 - \lambda_3^2\right)} \Bigg\{ & 2\sum_{i=1}^{3} f_1\left(m_{N_i}, m_{\nu_i}\right) + 6\sum_{quarks} m_q^2\, a^R(m_q) + \\
& +2\sum_{\ell=e\mu\tau} m_\ell^2\, a^R(m_\ell) - 3m_W^2\left[a^R(m_W) + \frac{2}{3}m_W^2\right] + \\
& -\frac{3}{2}m_Z^2\left[a^R(m_Z) + \frac{2}{3}m_Z^2\right] + \\
& +\Bigg[\frac{m_{\mathrm{I}}^2\, a^R(m_{\mathrm{I}})}{4\left(m_{\mathrm{I}}^2 - m_{\mathrm{II}}^2\right)\left(m_{\mathrm{I}}^2 + m_{\mathrm{II}}^2 - 2\lambda_1 w_H^2\right)} \times \\
& \qquad \times\Big(3m_{\mathrm{II}}^4 - 10\lambda_1 m_{\mathrm{II}}^2 w_H^2 + 8\lambda_1\left(\lambda_1 - \lambda_3\right) w_H^4 + \\
& \qquad\quad +2m_{\mathrm{I}}^2(m_{\mathrm{II}}^2 - 2(\lambda_1 - \lambda_3) w_H^2)\Big) + \\
& \quad +(m_{\mathrm{I}} \leftrightarrow m_{\mathrm{II}})\Bigg]\Bigg\},
\end{aligned}
\tag{163}
$$

$$
\begin{aligned}
v_{\varphi(1)} = \frac{\lambda_3}{(4\pi)^2 \left(\lambda_1\lambda_2 - \lambda_3^2\right) w_H^2 w_\varphi} \Bigg\{ & -2\sum_{i=1}^{3} f_2\left(m_{N_i}, m_{\nu_i}\right) - 6\sum_{quarks} m_q^2\, a^R(m_q) + \\
& -2\sum_{\ell=e\mu\tau} m_\ell^2\, a^R(m_\ell) + 3m_W^2\left[a^R\left(m_W\right) + \frac{2m_W^2}{3}\right] +
\end{aligned}
$$

$$+\frac{3}{2}m_Z^2\left[a^R\left(m_Z\right)+\frac{2m_Z^2}{3}\right]\Bigg\}+$$

$$-\Bigg\{\frac{a^R(m_{\rm I})}{2(4\pi)^2\lambda_2 m_{\rm II}^2\left(m_{\rm I}^2-m_{\rm II}^2\right)w_\varphi}\Big[\lambda_2\left(m_{\rm I}^2-2\lambda_1 w_H^2\right)\left(m_{\rm II}^2+4\lambda_1 w_H^2\right)+$$

$$+2\lambda_3\left(m_{\rm II}^2-2\lambda_1 w_H^2\right)\left(m_{\rm I}^2+m_{\rm II}^2-2\lambda_1 w_H^2\right)\Big]+$$

$$+(m_{\rm I}\leftrightarrow m_{\rm II})\Bigg\}. \tag{164}$$

The symbol $(m_{\rm I}\leftrightarrow m_{\rm II})$ used in the above formulae denotes a term obtained by interchanging the two masses, m_I and m_{II}, in the preceding one. The functions f_1 and f_2 expressing contributions of the neutrinos read

$$f_1(m_N,m_\nu)=\frac{1}{2\left(m_N+m_\nu\right)\lambda_2 w_\varphi^2}\times$$

$$\times\Big\{\left(\lambda_3\left(m_\nu-m_N\right)w_H^2+2\lambda_2 m_\nu w_\varphi^2\right)m_N^2\,a^R\left(m_N\right)+$$

$$+\left(\lambda_3\left(m_N-m_\nu\right)w_H^2+2\lambda_2 m_N w_\varphi^2\right)m_\nu^2\,a^R\left(m_\nu\right)\Big\},$$

$$f_2(m_N,m_\nu)=\frac{1}{2\left(m_N+m_\nu\right)\lambda_3 w_\varphi^2}\times$$

$$\times\Big\{\left(\lambda_1\left(m_\nu-m_N\right)w_H^2+2\lambda_3 m_\nu w_\varphi^2\right)m_N^2\,a^R\left(m_N\right)+$$

$$+\left(\lambda_1\left(m_N-m_\nu\right)w_H^2+2\lambda_3 m_N w_\varphi^2\right)m_\nu^2\,a^R\left(m_\nu\right)\Big\}.$$

The factors **a**, **b** and **c** parameterizing the matrix (73), obtained from the general expressions (42) and (45), read

$$\mathbf{a}=\sum_{i=1}^{3}f_{11}(m_{N_i},m_{\nu_i})-\frac{1}{4\pi^2 w_H^2}\left\{3\sum_{quarks}m_q^2\ln\frac{m_q}{\bar\mu}+\sum_{\ell=e\mu\tau}m_\ell^2\ln\frac{m_\ell}{\bar\mu}\right\}+\frac{\lambda_1}{16\pi^2}+$$

$$+\frac{3}{8\pi^2 w_H^2}\left\{2m_W^2\left[\ln\left(\frac{m_W}{\bar\mu}\right)-\frac{5}{12}\right]+m_Z^2\left[\ln\left(\frac{m_Z}{\bar\mu}\right)-\frac{5}{12}\right]\right\}+$$

$$-\frac{3\,m_{\rm I}^2\,m_Z^2\left(m_{\rm II}^2-2\lambda_1 w_H^2\right)}{8\pi^2 w_H^2\left(m_{\rm I}^2-m_{\rm II}^2\right)\left(m_{\rm I}^2-m_Z^2\right)}\ln\left(\frac{m_{\rm I}}{m_Z}\right)+$$

$$+\frac{3\,m_{\rm II}^2\,m_Z^2\left(m_{\rm I}^2-2\lambda_1 w_H^2\right)}{8\pi^2 w_H^2\left(m_{\rm I}^2-m_{\rm II}^2\right)\left(m_{\rm II}^2-m_Z^2\right)}\ln\left(\frac{m_{\rm II}}{m_Z}\right), \tag{165}$$

$$\mathbf{c}=\sum_{i=1}^{3}f_{22}(m_{N_i},m_{\nu_i})+\frac{\lambda_2}{16\pi^2}, \tag{166}$$

$$\mathbf{b} = \sum_{i=1}^{3} f_{12}(m_{N_i}, m_{\nu_i}) \,. \tag{167}$$

The functions f_{11}, f_{22} and f_{12} which represent contributions of the neutrino loops are given by

$$f_{11}(m_N, m_\nu) = \frac{1}{(4\pi)^2 w_H^2} \left\{ \frac{m_N m_\nu \left(m_N^2 - 4 m_N m_\nu + m_\nu^2\right)}{(m_N + m_\nu)^2} + \right.$$

$$\left. -4 m_N m_\nu \ln\left(\frac{m_N}{\bar{\mu}}\right) + \frac{4 m_N m_\nu^2 \left(m_\nu^3 - 2 m_N^3\right)}{(m_N - m_\nu)(m_N + m_\nu)^3} \log\left(\frac{m_\nu}{m_N}\right) \right\},$$

$$f_{22}(m_N, m_\nu) = \frac{1}{(4\pi)^2 w_\varphi^2} \left\{ \frac{m_\nu m_N \left(m_\nu^2 - 4 m_\nu m_N + m_N^2\right)}{(m_\nu + m_N)^2} + \right.$$

$$+ \frac{2 m_N^2 \left(m_\nu^4 + 4 m_\nu^2 m_N^2 - 2 m_\nu^3 m_N - m_N^4\right)}{(m_N - m_\nu)(m_\nu + m_N)^3} \ln\left(\frac{m_N}{\bar{\mu}}\right) +$$

$$\left. + \frac{2 m_\nu^2 \left(m_\nu^4 + 2 m_\nu m_N^3 - 4 m_\nu^2 m_N^2 - m_N^4\right)}{(m_N - m_\nu)(m_\nu + m_N)^3} \ln\left(\frac{m_\nu}{\bar{\mu}}\right) \right\},$$

$$f_{12}(m_N, m_\nu) = \frac{1}{(4\pi)^2 w_H\, w_\varphi} \left\{ \frac{4 m_\nu^2 m_N^2 \left(m_\nu^2 - m_\nu m_N + m_N^2\right)}{(m_N - m_\nu)(m_\nu + m_N)^3} \ln\left(\frac{m_\nu}{m_N}\right) + \right.$$

$$\left. - \frac{m_\nu m_N \left(m_\nu^2 - 4 m_\nu m_N + m_N^2\right)}{(m_\nu + m_N)^2} \right\}.$$

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